# ELEMENTARY COURSE

OF

# MATHEMATICS.

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AN

# ELEMENTARY COURSE

OF

# MATHEMATICS,

DESIGNED PRINCIPALLY FOR STUDENTS OF THE UNIVERSITY OF CAMBRIDGE.

BY THE

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FIFTH EDITION.

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"As Tennis is a game of no use in itself, but of great use in respect it maketh a quick eye, and a body ready to put itself into all postures; so in the Mathematics, that use which is collateral and intervenient is no less worthy than that which is principal and intended."

BACON, *Advancement of Learning.*

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LONDON: BELL AND DALDY.
1857.

## PREFACE TO THE FIFTH EDITION.

This Edition differs very slightly from that which preceded it. The improvements and alterations are such as to require no additional remarks; I therefore reprint the Preface to the Fourth Edition entire.

H. GOODWIN.

Cambridge.
*July*, 1857.

## PREFACE TO THE FOURTH EDITION.

The sale of three large impressions of this work renders it unnecessary for me to reprint in this Fourth Edition the preparatory remarks which were prefixed to the book at its first appearance. It will however be desirable that the student should have before him the schedule of subjects, as fixed by the Grace of the Senate which governs the first three days of the examination for Mathematical Honours, and which consequently determines what shall and what shall not form part of the following Mathematical Course.

The Grace is as follows:

1. That Questions and Problems being proposed to the Questionists on eight days, instead of six days as at present, the first three days be assigned to the more elementary, and the last five to the higher parts of

Mathematics: that after the first three days, there shall be an interval of eight days; and that on the seventh of these days the Moderators and Examiners shall declare, what persons have so acquitted themselves as to deserve Mathematical Honours.

2. That those who are declared to have so acquitted themselves, and no others, be admitted to the Examination in the higher parts of Mathematics: and that after that Examination, the Moderators and Examiners, taking into account the Examination of all the eight days, shall arrange all the Candidates who have been declared to deserve Mathematical Honours into the three classes of Wranglers, Senior Optimes, and Junior Optimes, as has been hitherto usual; and that these classes be published in the Senate-House at nine o'Clock on the Friday morning preceding the general B.A. Admission.

3. That the subjects of the Examination on the first three days shall be those contained in the following Schedule:—

Euclid. Book I. to VI. Book XI, Props. I. to XXI. Book XII, Props. I, II.

Arithmetic and the elementary parts of Algebra; namely, the Rules for the fundamental Operations upon Algebraical Symbols, with their proofs; the solution of simple and quadratic equations; Arithmetical and Geometrical Progression, Permutations and Combinations, the Binomial Theorem, and the principles of Logarithms.

The elementary parts of Plane Trigonometry, so far as to include the solution of triangles.

The elementary parts of Conic Sections, treated Geometrically, together with the values of the Radius of Curvature, and of the Chords of Curvature passing through the Focus and Centre.

The elementary parts of Statics, treated without the Differential Calculus; namely, the composition and resolution of Forces acting in one plane on a point, the Mechanical Powers, and the properties of the Centre of Gravity.

The elementary parts of Dynamics, treated without the Differential Calculus; namely, the Doctrine of Uniform and Uniformly accelerated

Motion, of falling Bodies, Projectiles, Collision, and Cycloidal Oscillations.

The 1st, 2nd, and 3rd Sections of NEWTON'S PRINCIPIA; the Propositions to be proved in Newton's manner.

The elementary parts of HYDROSTATICS, treated without the Differential Calculus; namely, the pressure of non-elastic Fluids, specific Gravities, floating Bodies, the pressure of the Air, and the construction and use of the more simple Instruments and Machines.

The elementary parts of OPTICS: namely, the laws of Reflexion and Refraction of Rays at plane and spherical surfaces, not including Aberrations; the Eye; Telescopes.

The elementary parts of ASTRONOMY; so far as they are necessary for the explanation of the more simple phænomena, without calculations.

4. That in all these subjects, Examples, and Questions arising directly out of the propositions, shall be introduced into the Examination, in addition to the propositions themselves.

5. (*This article refers merely to the days and hours of Examination, and is therefore omitted.*)

6. That the Moderators and Examiners shall be authorised to declare Candidates, though they have not deserved Mathematical Honours, to have deserved to pass for an Ordinary Degree, so far as the Mathematical part of the Examination for such degree is concerned; and such persons shall accordingly be excused the Mathematical part of the Examination for an Ordinary Degree, and shall only be required to pass in the other subjects, namely, in the parts of the Examination assigned in the Schedule to the last two days: but such excuse shall be available to such persons only for the Examination then in progress.

This volume is intended to contain, and I believe does contain, all that is necessary for the three days' examination according to the above schedule, with the exception of Euclid and Arithmetic.

The present edition is considerably improved and enlarged; the enlargement however is the result almost entirely of additional explanations, and illustrations, the number of new articles introduced is extremely small. Several mistakes into which I had inadvertently fallen in preceding editions have been corrected, and I desire to tender my thanks to those friends who have called my attention to errors which I had overlooked.

In the treatise on Astronomy I have endeavoured as much as possible to lead the student to consult other works in addition; the subject is so extensive as to be with difficulty brought into a small compass, not to mention the advantage of studying the works of those who are not merely writers on Astronomy but themselves Astronomers. With this view I have referred repeatedly to Sir J. F. W. Herschel's "Outlines", as well as Mr Airy's "Ipswich Lectures": I have also quoted for the same purpose from Humboldt's "Cosmos", and Buff's "Physics of the Earth"; I have likewise referred to the "Elementary Chapters in Astronomy" which I translated from Biot's *Astronomie Physique* and published some time since, and which as an introduction to Astronomy, clear in exposition and rendered interesting by historical details, appear to me unrivalled.

Notwithstanding the increase of size in the present edition, amounting to about *one hundred* pages of fresh matter, the publisher has found it to be possible (in consequence of the large circulation) to reduce the price of the work; this has accordingly been done with my full concurrence.

H. GOODWIN.

CAMBRIDGE.

*January*, 1853.

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# ALGEBRA.

# ALGEBRA.

1. ALGEBRA in its most comprehensive sense may be defined as being the science of reasoning by means of general written symbols.*

In its simplest form we may consider algebra as a more general species of arithmetic, in which the reasoning and the operations refer to certain general representatives of numbers, instead of being applied to the symbols of specific numbers, viz. the digits 1, 2, 3...which are the subject of arithmetical reasoning. The science of algebra is of a far more general and comprehensive character than this; but even restricting our views to such a science as would be formed by a substitution of symbols of numbers in general for the nine digits, we may see at once the greatness of the advance which we have made beyond the limits of mere arithmetic, because any rule which has been established by means of algebraical symbols will be universally true, since the symbols may represent any numbers whatever; whereas it is difficult to establish a rule by means of operations which deal with particular numbers only. This remark will be understood better as the student proceeds.

In the following treatise it will be assumed that the student has already made himself acquainted with arithmetic; but some of the rules and operations of that science will be proved and explained as applications of algebra. Indeed, the theory of some of the arithmetical processes will seldom be seen distinctly, until the mind has been informed by the more general science.

* The etymology of the name, Algebra, is given in various ways. It is, however, pretty generally considered, that the word is Arabian, and that from those people we had the name, as well as the art itself, as is testified by Lucas de Burgo, the first European author whose treatise was printed on this art, and who also refers to former authors and masters, from whose writings he had learned it. The Arabic name he gives it, is Alghebra e Almucabala, which is explained to signify the art of restitution and comparison, or opposition and restoration, or resolution and equation, all which agree well enough with the nature of this art. Some, however, derive it from various other Arabic words. Hutton's Tracts, *History of Algebra.*

2. The symbols used to denote *numbers* or *quantities* are usually the letters of the alphabet; and it is the common practice to express known or determined quantities by the early letters, as $a$, $b$, $c$..., and unknown by the latter, as $x$, $y$, $z$; but this rule is purely conventional, and need not be strictly followed.

It will be convenient here to enumerate and explain the various signs which are used in algebra.

3. + *Plus*, signifies that the quantity to which it is prefixed must be added. Thus $2+3$ is the same thing as $5$; using letters, $a+b$ represents the *sum* of $a$ and $b$, whatever are the values of $a$ and $b$, or $a$ and $b$ *added together*.

4. − *Minus*, signifies that the quantity to which it is prefixed must be subtracted. Thus $3-2$ is the same thing as $1$; and $a-b$ represents $a$ with $b$ taken from it.

5. Since the signs + and − prefixed to a quantity $b$ indicate addition and subtraction, they would seem to imply some antecedent quantity to which $b$ is to be added or from which it is to be subtracted; they are used however without this restriction, and for the present it will be sufficient for the student to consider $+b$ as a quantity which *is to be* added, and $-b$ as a quantity which *is to be* subtracted. One of the signs + and − is supposed to be prefixed to every algebraical quantity: $+a$ is termed a *positive* quantity, $-a$ a *negative* quantity. When $+a$ is not preceded by another quantity, it is usual for shortness' sake to omit the +, and to write simply $a$.

A simple illustration of the meaning of a negative quantity may here be of service. A *debt* may be regarded as a negative quantity, inasmuch as it is a quantity *to be subtracted* in case of there being any property, (which is *positive*,) from which to subtract it.

6. × *Into*, signifies that the quantities between which it stands are to be multiplied together: thus $2 \times 3$ is equivalent to 6.

This sign is frequently omitted, or its place supplied by a point: thus $a \times b$, $ab$, $a.b$, are equivalent.

7. When the same quantity is multiplied into itself several times, the product is represented in an abbreviated form, by placing above the quantity a figure indicating the number of times that it is repeated: thus $a \times a$ is written $a^2$, and $a \times a \times a$, $a^3$: $a^2$ is called the second *power*, or the square of $a$; $a^3$ the third *power*, or the cube; $a^4$ the fourth power, and so on: $a^1$ is the same thing as $a$.

The figure which indicates the *power* of a quantity is called its *index* or *exponent*. The meaning of the terms *power, index, exponent*, will be hereafter much extended. (See Art. 25.)

8. $\div$ *Divided by*, signifies that the former of two quantities between which it is placed is to be divided by the latter. Thus $6 \div 2$ is equivalent to 3.

Division is however more generally represented by writing the two quantities as a vulgar fraction: thus $a \div b$ is written $\frac{a}{b}$, and is commonly read for shortness' sake thus, *a by b*.

We will anticipate the result of a subsequent article, (Art. 25,) by saying that $\frac{1}{a^n}$ may also be written thus, $a^{-n}$.

9. The *difference* of two quantities is sometimes represented by the sign $\sim$: thus $a \sim b$ means $a - b$, or $b - a$, according as $a$ is greater or less than $b$.

10. When several quantities are enclosed in a *bracket*, thus $(a + b - c)$, it is intended that any sign prefixed or affixed to the bracket should apply to all the quantities included by it: thus $a - (b + c)$ means that $b$ and $c$ are both to be subtracted from $a$, $(a + b)^2$ means that the sum of $a + b$ is to be multiplied by itself, $(2 + 1)(3 + 2)$ is equivalent to $3 \times 5$ or 15.

A *vinculum* or line drawn over several quantities, thus $\overline{a + b + c}$, is in all respects equivalent to a bracket.

11. $\therefore$ is an abbreviation for the word *therefore*; and $\because$ for the word *because*.

12. The *square root* of a quantity is represented by writing over the quantity the sign $\sqrt[2]{\phantom{x}}$ or more briefly $\sqrt{\phantom{x}}$,

which is in fact a corruption of the letter *r* standing for *radix* or *root*, and is termed a *radical:* thus $\sqrt[2]{a}$ or $\sqrt{a}$ means the square root of $a$. Similarly the *cube root* is denoted by $\sqrt[3]{a}$, the *fourth* root by $\sqrt[4]{a}$, and so on.

We will again anticipate the result of an article already referred to, (Art. 25) by stating that $\sqrt[n]{a}$ may also be written thus, $a^{\frac{1}{n}}$; in like manner $\sqrt[n]{a^m}$ may be written thus, $a^{\frac{m}{n}}$.

A quantity under a *radical* sign, the root of which cannot be extracted, is called an *irrational* quantity or a *surd:* thus $\sqrt{3}$ is a surd quantity. Quantities which involve no surds are called *rational.*

13. The number or quantity by which any other quantity is multiplied, is frequently called its *coefficient:* thus in the quantities $ax$, $7y$, $a$ and $7$ may be called the coefficients of $x$ and $y$ respectively. When no coefficient is prefixed to a letter, 1 is always understood.

14. Any combination of symbols is called an algebraical *expression*, and sometimes an algebraical *formula.*

An expression is said to be of $n$ dimensions with respect to any letter, when the highest power of that letter in the expression is $n$: thus $a^3 + 2a^2 + 1$ is of *three* dimensions with respect to $a$.

When an expression is composed of *two* quantities connected by the sign + or −, it is called a *binomial* expression; when of *three*, a *trinomial;* when of several, a *polynomial:* $a + b$ is a *binomial*, $a + b + c + d$ a *polynomial.*

When an expression is composed of quantities connected by the sign ×, (either expressed or understood,) the several quantities are called *factors* of the expression: thus $a$, $b$, $c$, are *factors* of $abc$.

15. = *Equals*, signifies that the quantities between which it is placed are equal to each other. Thus the symbolical sentence $2 + 3 = 5$ expresses the fact that 2 and 3 added together make 5.

The signs > and < are sometimes used to denote respectively *greater than* and *less than.* Thus $a > b$ would express that $a$ is greater than $b$.

16. An algebraical sentence expressing equality between two expressions, such as $x + 1 = 2$, or $x^2 + x + 1 = 0$, is called an *equation*.

17. *Similar* or *like* algebraical quantities are such as differ only in the value of their numerical coefficient: thus $2a$ and $5a$ are *like* quantities.

18. One quantity is said to be a *multiple* of another, when the one can be divided by the other without remainder, or when the one contains the other as a *factor:* thus $4a$ is a multiple of $a$, and $ax$ of $x$.

19. One quantity is said to be a *measure* of another, when the former will divide the latter without remainder.

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20. The student will find it advantageous, before proceeding further, to fix in his mind the knowledge acquired from the preceding articles, by practising upon some simple examples. And it may be worth while in this place to remark, that the science of algebra, and others which will be treated of hereafter, are most easily to be acquired by the practice of them; and indeed it may be said to be almost impossible to acquire and retain a perfect familiarity with algebraical theorems, except through the medium of a considerable amount of industry expended on the working out of examples. Wherefore, once for all, the student is earnestly requested after reading any new rule or theorem to turn to the examples illustrative of it, and work as many as possible before proceeding further.

## ADDITION.

21. Rule. *The addition of algebraical quantities is performed by connecting those that are unlike with their proper signs, and collecting those that are like into one sum.*

The addition of algebraical quantities would in fact be performed by writing them one after another, and connecting them by the sign +; but the preceding rule indicates the mode of reducing the sum so written down to its simplest form.

When *like* quantities occur with different signs, their algebraical sum is found by taking the smaller coefficient from the greater and prefixing the sign of the greater: thus $4a-3a$ would be written $a$, and $-4a+3a$ would be written $-a$.

It will be seen that addition thus considered is a very different operation, in some respects, from arithmetical addition, since in arithmetic to *add* is always to *increase*, but in algebra to *add* is only to *connect a series of quantities with their proper signs;* and a quantity may therefore be decreased by having another, which is negative, added to it.

The order in which the result of the addition of several quantities is written down is immaterial, but it is usual *cœteris paribus* to follow the order of the alphabet: thus we should write $a+b+c$, not $a+c+b$.

The following are examples of addition, which the student may verify:

$$\begin{array}{rr}
 & 2a+b \\
 & a+3b \\
\hline
\text{Sum} & 3a+4b
\end{array}
\qquad
\begin{array}{r}
-2a+b \\
a+3b \\
\hline
-a+4b
\end{array}
\qquad
\begin{array}{r}
2a-b \\
a+3b \\
\hline
3a+2b
\end{array}
\qquad
\begin{array}{r}
2a-b \\
a-3b \\
\hline
3a-4b
\end{array}$$

$$\begin{array}{r}
a^2+2ab+3b^2 \\
-2a^2-ab+4b^2 \\
3a^2+3ab-2b^2 \\
\hline
2a^2+4ab+5b^2
\end{array}
\qquad
\begin{array}{r}
-a+b+c+d \\
a-b+c+d \\
a+b-c+d \\
a+b+c-d \\
\hline
2a+2b+2c+2d
\end{array}$$

$$\begin{array}{l}
a^2+bc+cd \\
b^2+bd+c^2 \\
a^2+b^2+c^2 \\
\hline
2a^2+2b^2+2c^2+bc+bd+cd
\end{array}$$

$$\begin{array}{l}
ax+by+cz \\
bx-cy-az \\
-cx+ay-bz \\
\hline
(a+b-c)x+(b-c+a)y+(c-a-b)z
\end{array}$$

## SUBTRACTION.

22. Rule. *One algebraical quantity may be subtracted from another by changing its sign and adding it to the other.*

The reason of this rule is apparent; for to subtract $+a$ is by definition the same thing as to add $-a$; and to subtract a quantity which itself ought to be subtracted is nothing else than to add that quantity. Using an illustration already adopted, we may say that to *subtract* a *debt* is the same thing as to *add* to *property*.

Hence also we see that $-$ prefixed to a bracket changes the sign of all the quantities in that bracket, so that $-(a+b-c)$ is equivalent to $-a-b+c$.

### EXAMPLES.

| | | | |
|---|---|---|---|
| | $2a+b$ | $2a-b$ | $a+b-2c$ |
| | $a+3b$ | $-a+3b$ | $a-b+c$ |
| Difference | $a-2b$ | $3a-4b$ | $2b-3c$ |

| | | |
|---|---|---|
| $2a^2+3ab-b^2$ | $a+b-(c+d)$ | $(a+b)x^2-(c-d)y^2$ |
| $a^2-4ab+3b^2$ | $a-(b-c)+d$ | $(b+c)x^2+(a+d)y^2$ |
| $a^2+7ab-4b^2$ | $2b-2c-2d$ | $(a-c)x^2-(a+c)y^2$ |

## MULTIPLICATION.

23. *Unlike* quantities cannot be multiplied together any further than by connecting them by the sign $\times$, or writing them together without sign. Thus $a$ multiplied by $b$ gives $ab$ or $ba$, for it is indifferent whether we consider $a$ to be multiplied by $b$ or $b$ by $a$.

But *like* quantities are multiplied together by *adding their indices*. Thus $a^3 \times a^4 = a^7$, for $a^3$ signifies $aaa$ and $a^4$ signifies $aaaa$, therefore $a^3 \times a^4 = aaa \times aaaa = aaaaaaa = a^7$ by definition, for $a^7$ means nothing else but $a$ multiplied into itself *seven* times.

The *sign* of the product of two quantities is determined by this rule; viz. the product of two quantities affected by the *same* sign is *positive*, of two affected by *different* signs *negative*.

$$\text{Thus } +a \times +b = +ab;$$
$$-a \times +b = -ab;$$
$$+a \times -b = -ab;$$
$$-a \times -b = +ab.$$

This rule may be thus explained:

(1) $+a \times +b$ signifies that $a$ is to be added $b$ times, which is the same as adding unity $ab$ times, therefore the result is $+ab$.

(2) $+a \times -b$ signifies that $b$ is to be subtracted $a$ times, which is the same thing as subtracting unity $ab$ times, therefore the result is $-ab$.

(3) $-a \times +b$ signifies that $a$ is to be subtracted $b$ times, therefore, as in the last case, the result is $-ab$.

(4) $-a + -b$ may be interpreted to mean that $-a$ is to be subtracted $b$ times, or that $-ab$ is to be *subtracted*, or that $ab$ is to be *added*, therefore the result is $+ab$.

Numbers are multiplied together as in common arithmetic: thus $2a \times 3b$ is not written $2 \times 3ab$ but $6ab$.

What has been said hitherto applies chiefly to the multiplication of simple algebraical quantities, or expressions of *one* term only. When two polynomials are multiplied together, each term in the multiplicand must be multiplied by each term in the multiplier, and the sum of all such products (arranged as is most convenient) will be the complete product required.

It is usual in algebraical multiplication to commence with the term on the left hand of an expression, instead of commencing on the right as in arithmetic.

When the same letter occurs in an expression with different indices, it is usual, and in most cases of the application

of algebra necessary, to arrange the expressions according to the powers of that letter: thus the expression $1 - 3x + 3x^2 - x^3$ ought not to be written $1 + 3x^2 - 3x - x^3$, but it may with propriety be written as we have given it, in which case it is said to be arranged according to *ascending powers of $x$;* or it may be written thus $-x^3 + 3x^2 - 3x + 1$, in which case it is said to be arranged according to *descending powers of $x$.* The student cannot be too careful in attending to the proper *arrangement* of expressions.

EXAMPLES.

$$\begin{array}{rl} & a+b \\ & c+d \\ \hline & ac+bc \\ & \qquad ad+bd \\ \hline \text{Product} & ac+bc+ad+bd \\ \hline \end{array} \qquad\qquad \begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \qquad -ab-b^2 \\ \hline a^2 \qquad\quad -b^2 \\ \hline \end{array}$$

$$\begin{array}{l} a+\ bx+\ cx^2 \\ a-\ bx+\ cx^2 \\ \hline a^2+abx+acx^2 \\ \quad -abx-b^2x^2-bcx^3 \\ \qquad\qquad acx^2+bcx^3+c^2x^4 \\ \hline a^2 \qquad +(2ac-b^2)x^2+c^2x^4 \\ \hline \end{array}$$

In this example the product has been arranged according to *ascending powers of $x$,* because the multiplicand and multiplier were so arranged; and on this account the two terms involving $x^2$, viz. $2acx^2$ and $-b^2x^2$, have been collected into one term, and the combined quantity $2ac - b^2$ is considered as the coefficient of $x^2$.

$$\begin{array}{l} x^2+2x-1 \\ x^2-2x+1 \\ \hline x^4+2x^3-x^2 \\ \quad -2x^3-4x^2+2x \\ \qquad\qquad x^2+2x-1 \\ \hline x^4 \qquad -4x^2+4x-1 \\ \hline \end{array}$$

The operation of multiplication may be sometimes abbreviated in the following manner.

It is easily seen by actual multiplication, that

$$(a+b)\times(a-b)=a^2-b^2,$$

or that *the product of the sum and difference of two quantities is equal to the difference of their squares;* a theorem which may be used in the multiplication of such quantities as those in the last example, or more generally, in the multiplication of two polynomials which involve the same algebraical quantities but different algebraical signs.

Thus $(x^2+2x-1)\times(x^2-2x+1)=(x^2+\overline{2x-1})\times(x^2-\overline{2x-1})$

$$=x^4-(2x-1)^2=x^4-(4x^2-4x+1)=x^4-4x^2+4x-1,$$

the same result as before.

Again, $(x+y+z-u)\times(x-y+z+u)$

$$=(\overline{x+z}+\overline{y-u})\times(\overline{x+z}-\overline{y-u})$$

$$=(x+z)^2-(y-u)^2=x^2+2xz+z^2-y^2+2yu-u^2,$$

a result which may be verified by direct multiplication.

## DIVISION.

**24.** Division being the inverse of multiplication, its rules may be deduced from those of multiplication.

Quantities which are not powers of the same quantity can be divided one by another only by writing one under the other in the form of a vulgar fraction.

But quantities which are powers of the same quantity can be divided one by another by subtracting the index of the divisor from that of the dividend. Thus $a^3\div a=a^2$, because, as we have seen, $a\times a^2=a^3$.

The rule of signs is this; the division of quantities of *like* signs gives a *positive* quantity, and of *unlike* signs a negative.

Sometimes the division of unlike quantities can be partially effected: thus $\frac{a^2b^2c}{abd} = \frac{abc}{d}$, where the dividend can be divided by the factors $a$ and $b$ of $abd$, but not by the factor $d$.

When a polynomial is to be divided by a simple quantity, each term of the polynomial must be divided by it, and the sum of the terms so found affected with their proper signs will be the quotient.

The process of dividing one polynomial by another is one of greater difficulty, but is rendered sufficiently simple by its analogy to long division in common arithmetic. The first step is to arrange the divisor and dividend according to either ascending or descending powers of some letter common to the two; the division of the first term of the dividend by the first term of the divisor gives the first term of the quotient; multiply the divisor by this term, and subtract the product from the dividend; bring down as many more of the terms of the dividend as may be required, and repeat the process until all the terms have been brought down.

The only point in this rule which seems to require explanation, is the arranging of the expressions according to powers of some common letter. The reason may be given thus: divison is the inverse of multiplication, and in order to make division successful we must be sure that we follow exactly the reverse steps of some particular mode of multiplying, for two expressions may be multiplied together in many different ways according to the arrangement which we choose to adopt; now we are sure of following an exactly reverse process by attending to the rule of arrangement which has been given, for the quotient and divisor may be conceived to have been multiplied together according to this rule to form the dividend.

It cannot be positively asserted, that the operation of division will never succeed unless the quantities be arranged according to the powers of some common letter; but it may be said that in general the operation will fail, and that we can never ensure success unless the quantities be so arranged.

EXAMPLES.

$$\begin{array}{l}
a+b)\ a^2-b^2\ (a-b \quad \text{Quotient} \\
\qquad\quad a^2+ab \\
\qquad\quad \overline{\quad -ab-b^2} \\
\qquad\qquad -ab-b^2 \\
\end{array}$$

$$\begin{array}{l}
x-y)\ x^3-3x^2y+3xy^2-y^3\ (x^2-2xy+y^2 \\
\qquad\quad x^3-x^2y \\
\qquad\quad \overline{\quad -2x^2y+3xy^2} \\
\qquad\qquad -2x^2y+2xy^2 \\
\qquad\qquad \overline{\qquad\qquad xy^2-y^3} \\
\qquad\qquad\qquad\qquad xy^2-y^3 \\
\end{array}$$

$$\begin{array}{l}
x+y)\ x^n+y^n\ (x^{n-1}-x^{n-2}y+x^{n-3}y^2-\&c. \\
\qquad\quad x^n+x^{n-1}y \\
\qquad\quad \overline{\quad -x^{n-1}y+y^n} \\
\qquad\qquad -x^{n-1}y-x^{n-2}y^2 \\
\qquad\qquad \overline{\qquad\qquad x^{n-2}y^2+y^n} \\
\qquad\qquad\qquad\qquad x^{n-2}y^2+x^{n-3}y^3 \\
\qquad\qquad\qquad\qquad \overline{\qquad\qquad -x^{n-3}y^3+y^n} \\
\qquad\qquad\qquad\qquad\qquad\qquad \&c. \quad \&c.
\end{array}$$

Let us determine under what circumstances the preceding division will terminate. Suppose $n$ to be an odd number, then we shall arrive at length at a remainder $xy^{n-1}+y^n$, which is evidently divisible by $x+y$, and therefore the division terminates; but if $n$ be even, we shall have the remainder $-xy^{n-1}+y^n$, which is not divisible by $x+y$, and therefore the division does not terminate.

In like manner it may be shewn that $x^n+y^n$ is never divisible by $x-y$, and that $x^n-y^n$ is divisible by $x+y$ if $n$ is even, and by $x-y$ whether $n$ is even or odd.

The following example is given to shew the importance of arrangement:

$$a+1)\ a^2+2a+1\ (a+1$$
$$\underline{a^2+a}$$
$$a+1$$
$$\underline{a+1}$$

thus the operation terminates; but suppose we had proceeded thus:

$$a+1)\ 1+2a+a^2\left(\frac{1}{a}-\frac{1}{a^2}+ \text{\&c.}\right.$$
$$\underline{1+\frac{1}{a}}$$
$$-\frac{1}{a}+2a$$
$$\underline{-\frac{1}{a}-\frac{1}{a^2}}$$
$$\frac{1}{a^2}\ \text{\&c.}$$

and the operation will never come to an end.

Let us determine under what circumstances $x^2+ax+b$ is divisible by $x+y$.

$$x+y)\ x^2+ax+b\ (x+a-y$$
$$\underline{x^2+yx}$$
$$(a-y)x+b$$
$$\underline{(a-y)x+ay-y^2}$$
$$b-ay+y^2$$

We have then in general a remainder $b-ay+y^2$; and $x^2+ax+b$ is therefore not divisible by $x+y$, unless $y$ be such a quantity that $b-ay+y^2=0$.

In like manner we can determine the condition of $x^3 + ax^2 + bx + c$ being divisible by $x + y$.

$$
\begin{array}{l}
x + y)\ x^3 + ax^2 + bx + c\ (x^2 + (a - y)\,x + b - ay + y^2 \\
\qquad\quad \underline{x^3 + yx^2} \\
\qquad\qquad\quad (a - y)\,x^2 + bx \\
\qquad\qquad\quad \underline{(a - y)\,x^2 + (ay - y^2)\,x} \\
\qquad\qquad\qquad\qquad\qquad (b - ay + y^2)\,x + c \\
\qquad\qquad\qquad\qquad\qquad \underline{(b - ay + y^2)\,x + by - ay^2 + y^3} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad c - by + ay^2 - y^3
\end{array}
$$

Hence the division is not possible, unless $y$ be such a quantity that $c - by + ay^2 - y^3 = 0$.

## ON THE MEANING OF FRACTIONAL AND NEGATIVE INDICES.

25. Hitherto $a^n$ has been understood to signify $a \times a \times \ldots$ $n$ times, and therefore $n$ has been supposed to be a whole number. But it becomes a question whether it may not be possible to assign a meaning to the symbol $a^n$ in other cases, whether, for instance, we may not assign a meaning to $a^{\frac{1}{2}}$, and to $a^{-2}$. In doing so the only thing to be attended to is, that no supposition be made contradictory to any thing which we have at present laid down, and it will manifestly be most convenient that the rules for multiplying and dividing such quantities as we speak of should be the same as in the case of a positive index. Suppose then we make this convention, that the rule of indices which has been proved in the case of positive integer indices (Art. 23) shall hold true in all cases; that is, let us *assume* that *universally*,

$$a^m \times a^n = a^{m+n} \qquad (1)$$

$$\text{and} \quad \frac{a^m}{a^n} = a^{m-n} \qquad (2)^*;$$

then it will be found that these assumptions, which contradict

* It will be easily seen that these two assumptions are not independent, (2) being immediately deducible from (1).

nothing preceding them, will be sufficient to determine the meaning of $a^n$ when $n$ is fractional or negative. For we have by (1)

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a,$$

$$\text{but } a^{\frac{1}{2}} \times a^{\frac{1}{2}} = (a^{\frac{1}{2}})^2;$$

$$\therefore (a^{\frac{1}{2}})^2 = a,$$

$$\text{or } a^{\frac{1}{2}} = \sqrt{a}.$$

Or more generally,

$$a^{\frac{1}{p}} \times a^{\frac{1}{p}} \times \ldots\ldots p \text{ times} = a^{\frac{1}{p}+\frac{1}{p}\ldots\ldots\text{to } p \text{ terms}} = a;$$

$$\text{but } a^{\frac{1}{p}} \times a^{\frac{1}{p}} \times \ldots\ldots p \text{ times} = \left(a^{\frac{1}{p}}\right)^p;$$

$$\therefore \left(a^{\frac{1}{p}}\right)^p = a,$$

$$\text{or } a^{\frac{1}{p}} = \sqrt[p]{a}.$$

Hence the symbol $a^{\frac{1}{p}}$ represents the $p^{\text{th}}$ root of $a$, and $a^{\frac{q}{p}}$ represents the $p^{\text{th}}$ root of $a^q$: thus $4^{\frac{1}{2}} = 2$, and $4^{\frac{3}{2}} = 8$. It will be seen also that $\sqrt{\sqrt{a}} = \sqrt{a^{\frac{1}{2}}} = a^{\frac{1}{2}\times\frac{1}{2}} = a^{\frac{1}{4}}$, and so on.

Again, suppose that in (1) we write $-n$ for $n$, then we have

$$a^m \times a^{-n} = a^{m-n};$$

but by (2) $$\frac{a^m}{a^n} = a^{m-n};$$

$$\therefore a^m \times a^{-n} = \frac{a^m}{a^n},$$

which proves, that to multiply by $a^{-n}$ is the same thing as to divide by $a^n$, or that $a^{-n} = \frac{1}{a^n}$; thus $2^{-\frac{1}{2}} = \frac{1}{\sqrt{2}}$, and $4^{-\frac{1}{2}} = \frac{1}{2}$.

Thus we have assigned to negative and fractional indices a meaning not inconsistent with any thing which precedes, and which will be found of great service. We shall therefore henceforth use the symbols $\sqrt[p]{a}$, $\frac{1}{a^n}$, and $a^{\frac{1}{p}}$, $a^{-n}$ indifferently.

26. A rather remarkable consequence follows from (2), which will require a few words. If we suppose $m$ and $n$ to be equal, we have

$$\frac{a^m}{a^m} = a^{m-m},$$

$$\text{or } 1 = a^0.$$

This result is, at first sight, somewhat paradoxical; nevertheless it must be received as true, being a legitimate deduction from our previous assumptions; moreover, it is not wholly incapable of being interpreted in such a way as to make it intelligible. For suppose, *first*, that $a$ is a number greater than 1; then, if we extract its square root, it is evident that the square root will be less than the number itself; let the square root be again extracted and the number will be still further decreased; let this process be repeated a great many times, say a thousand, then the number will have decreased at each process, but will never be made less than 1, because if so, conversely a quantity less than 1 might be made greater than 1 by squaring, which is absurd: hence $a^{\frac{1}{2^{1000}}}$, which represents the square root of $a$ taken a thousand times, must be very nearly $= 1$, but not quite. Again, *secondly*, let $a$ be a number less than 1, and let the same process be performed upon it, then it is clear from the same kind of reasoning that the number will increase at each process, but that it can never become quite $= 1$: hence also in this case $a^{\frac{1}{2^{1000}}}$ nearly $= 1$, but not quite. But $\frac{1}{2^{1000}}$ is a very small quantity indeed, and it therefore appears that, whether $a$ be greater or less than 1, when $n$ is *very small*, $a^n$ *very nearly* $= 1$, and hence we can see the meaning of the equation $a^0 = 1$.*

* The meaning of the equation $a^0=1$, and of negative indices, may perhaps receive illustration thus. Take any power of $a$, as $a^6$, and divide it by $a$, the quotient again by $a$, and so on; then we shall produce the following series of quantities,

$$a^6, a^5, a^4, a^3, a^2, a^1, 1, \frac{1}{a}, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4}, \ldots\ldots\ldots (A).$$

Now take the same quantity $a^6$, and produce a series of quantities by continually diminishing its index by unity; the series will be

$$a^6, a^5, a^4, a^3, a^2, a^1, a^0, a^{-1}, a^{-2}, a^{-3}, a^{-4}, \ldots\ldots\ldots (B).$$

## THE GREATEST COMMON MEASURE.

27. DEF. *The greatest common measure of two numbers is the greatest number which will divide both without remainder.*

The greatest common measure of two algebraical expressions must be defined rather differently.

DEF. *Let two algebraical expressions be arranged according to descending powers of some common letter, then the factor of the highest dimension with respect to that letter, which divides both without remainder, is the greatest common measure of the two expressions.*

It would be more correct to speak of the *highest common divisor*, since the terms *greater* or *less* are not applicable to algebraical expressions, which are great or small according to the numerical values which we choose to assign to the letters involved: but, in accordance with established usage, the name of *greatest common measure* will be used.

28. *To investigate a rule for finding the greatest common measure of two algebraical expressions.*

Let $A$ and $B$ be two expressions arranged according to descending powers of some common letter, and let the highest power of that letter in $B$ be not higher than the highest in $A$. Divide $A$ by $B$, make the remainder the divisor and $B$ the dividend, and so on, until you come to a quantity which will divide without remainder; this last divisor will be the greatest common measure required.

The series ($A$) and ($B$) ought to be the same, since the commencements are the same, and the two are formed throughout by a uniform process; but this identity cannot subsist unless we have

$$a^0 = 1,\ a^{-1} = \frac{1}{a},\ a^{-2} = \frac{1}{a^2},\ a^{-3} = \frac{1}{a^3},\ a^{-4} = \frac{1}{a^4},\ \&c.$$

The meaning of fractional indices might be illustrated in a similar manner; but the view given in the text, respecting the ground upon which the theory of both fractional and negative indices is built, appears to be the most logical as well as the most simple which can be proposed.

The operation indicated may be represented as under,

$$\begin{array}{l}
B)\ A\ (p \\
\quad \underline{pB} \\
\quad C)\ B\ (q \\
\qquad \underline{qC} \\
\qquad D)\ C\ (r \\
\qquad\quad \underline{rD} \\
\qquad\quad\ \ \underline{0}
\end{array}$$

from which we have the following relations,

$$A = pB + C \qquad (1),$$

$$B = qC + D \qquad (2),$$

$$C = rD \qquad (3).$$

Now it is manifest that any quantity ($P$) which measures two others, $Q$, $R$, will measure a quantity such as $mQ \pm Nr$*, since it will divide each of the terms $mQ$ and $nR$†; but from (3) we see that $D$ measures $C$, therefore it measures $qC + D$, that is, it measures $B$, by (2); therefore it measures $pB + C$, that is, it measures $A$, by (1). Hence $D$ is a common measure of $A$ and $B$. It is also the *greatest* common measure; for if not, let it be $D'$; then since $D'$ measures both $A$ and $B$ it measures $A - pB$, that is, it measures $C$, by (1); therefore it measures $B - qC$, that is, it measures $D$, by (2): but $D'$ cannot measure $D$ if it be a quantity of higher dimensions, therefore $D'$ is not a greater common measure than $D$, that is, $D$ is the greatest common measure.

In order to render the preceding process successful, it will be necessary to modify the remainders, and also the given expressions, in such a manner as to avoid fractional coefficients in all the terms which occur. We may do this by multiplying by any quantity which does not introduce a new common measure, since it is clear that the proof which has been given

* The expression $mQ \pm nR$ stands for $mQ + nR$ and $mQ - nR$: it is read thus, $mQ$ *plus or minus* $nR$.

† If this does not appear manifest, the proof is as follows: Since $P$ measures $Q$, let $Q = qP$, and since $P$ measures $R$, let $R = rP$; therefore

$$mQ \pm nR = mqP \pm nrP = (mq \pm nr)\,P,$$

that is, $P$ measures $mQ \pm nR$.

will not be affected by supposing the expressions so modified. Also, any factor which is found to belong to one remainder, and not to the other which is used as the divisor or dividend to it, should be omitted.

The rule for finding the greatest common measure of two *numbers* follows at once from the preceding investigation; in the arithmetical process there is clearly no need of that modification of the remainders or the given quantities, which forms so important a part of the algebraical.

Ex. Find the greatest common measure of

$$3x^4 + 2x^3 + 3x + 2 \text{ and } 4x^3 + 10x^2 + 4x - 2.$$

The first thing to be done is to reject the factor 2, which belongs to the latter quantity and not to the former; then the operation is continued thus:

$$
\begin{array}{r|l}
 & 3x^4 + 2x^3 + 3x + 2 \\
 & 2 \\
\hline
2x^3 + 5x^2 + 2x - 1 & 6x^4 + 4x^3 + 6x + 4 \quad (3x \\
 & 6x^4 + 15x^3 + 6x^2 - 3x \\
\hline
 & -11x^3 - 6x^2 + 9x + 4 \\
 & 2 \\
\hline
 & -22x^3 - 12x^2 + 18x + 8 \quad (-11 \\
 & -22x^3 - 55x^2 - 22x + 11 \\
\hline
 & 43x^2 + 40x - 8 \\
\end{array}
$$

$$
\begin{array}{r|l}
 & 2x^3 + 5x^2 + 2x - 1 \\
 & 43 \\
\hline
43x^2 + 40x - 8 & 86x^3 + 215x^2 + 86x - 43 \quad (2x \\
 & 86x^3 + 80x^2 - 16x \\
\hline
 & 135x^2 + 92x - 43 \\
 & 43 \\
\hline
 & 5805x^2 + 3956x - 1849 \quad (135 \\
 & 5805x^2 + 5400x - 405 \\
\hline
 & -1444x - 1444 \\
\end{array}
$$

(Rejecting the factor $-1444$)

$$\begin{array}{r|l} x+1) & 43x^2 + 40x - 3 \ (43x - 3 \\ & \underline{43x^2 + 43x} \\ & \quad - 3x - 3 \\ & \quad \underline{- 3x - 3} \end{array}$$

Hence $x+1$ is the greatest common measure required.

The example here given is one of considerable complication, but is worthy of attention as illustrating the peculiar difficulties besetting the search for the greatest common measure. The student is particularly advised to obtain facility in working examples under this rule before proceeding further; not so much because the process of finding the greatest common measure is one of frequent use in practice, as because the greatest care is necessary to ensure the success of the operation, and the working of examples under this rule is therefore an excellent means of gaining that skill in the management of symbols which is essential in the subsequent applications of algebra.

It not unfrequently happens that the greatest common measure of two polynomials can be discovered without performing the operation above described: as for example, suppose it were required to find the greatest common measure of $x^4 - a^4$ and $x^3 - bx^2 + a^2x - a^2b$; we have,

$$x^4 - a^4 = (x^2 - a^2)(x^2 + a^2),$$

$$\text{and } x^3 - bx^2 + a^2x - a^2b = x^2(x-b) + a^2(x-b),$$

$$= (x^2 + a^2)(x - b),$$

and it is evident that $x^2 + a^2$ is the greatest common measure. In fact, the rule which we have given for discovering the greatest common measure may be dispensed with, whenever it is possible to ascertain by inspection the simple factors of which the polynomials are composed.

29. The greatest common measure of three quantities $A$, $B$, and $C$, is found thus: Find $D$ the greatest common measure of $A$ and $B$, then the greatest common measure of $D$ and $C$ will be the greatest common measure required.

For every measure of $A$ and $B$ measures $D$, and therefore every measure of $A$, $B$ and $C$ measures $C$ and $D$, and hence the highest measure of $A$, $B$, and $C$ will be the highest measure of $C$ and $D$.

Ex. Find the greatest common measure of $x^3+x^2-4x+2$, $x^3-x^2+x-1$, and $x^2+3x-4$.

First, to find the greatest common measure of $x^3+x-4x^2+2$ and $x^3-x^2+x-1$.

$$
\begin{array}{r|l}
x^3+x^2-4x+2 & x^3-x^2+x-1 \;(1 \\
 & \underline{x^3+x^2-4x+2} \\
 & \quad -2x^2+5x-3 \\
2x^2-5x+3 & 2x^3+2x^2-8x+4 \;(x \\
 & \underline{2x^3-5x^2+3x} \\
 & \quad 7x^2-11x+4 \\
 & \quad \underline{2} \\
 & \quad 14x^2-22x+8 \;(7 \\
 & \quad \underline{14x^2-35x+21} \\
 & \qquad 13x-13
\end{array}
$$

Rejecting the factor 13, we have for the new divisor $x-1$;

$$
\begin{array}{r|l}
x-1 & 2x^2-5x+3 \;(2x-3 \\
 & \underline{2x^2-2x} \\
 & \quad -3x+3 \\
 & \quad \underline{-3x+3}
\end{array}
$$

and $x-1$ is the greatest common measure of the first two given quantities.

We have now to find the greatest common measure of $x-1$ and $x^2+3x-4$.

$$
\begin{array}{r|l}
x-1 & x^2+3x-4 \;(x+4 \\
 & \underline{x^2-x} \\
 & \quad 4x-4 \\
 & \quad \underline{4x-4}
\end{array}
$$

$x-1$ is itself the greatest common measure required, and therefore, according to the preceding rule, is the greatest common measure of the three given quantities.

30. One application of the rule for finding the greatest common measure of two quantities is to the simplification of fractions; they may frequently be simplified by inspection, but, if this cannot be done, find the greatest common measure of the numerator and denominator by the preceding method and divide both numerator and denominator by it.

## THE LEAST COMMON MULTIPLE.

31. DEF. *If* A *and* B *are two algebraical expressions arranged according to descending powers of some common letter, and* M *the quantity of lowest dimensions with respect to that letter which is divisible by both expressions, then* M *is the least common multiple of* A *and* B.

To find $M$, let $m$ be any multiple of $A$ and $B$, so that

$$m = pA = qB;$$

then by definition $M$ will be that value of $m$ for which $p$ and $q$ are of the *lowest* dimensions. But since $pA = qB$, we have $\frac{p}{q} = \frac{B}{A}$, and therefore the proper value of $\frac{p}{q}$ will be found by reducing $\frac{B}{A}$ to its *lowest* terms. Hence if $D$ be the greatest common measure of $A$ and $B$,

$$p = \frac{B}{D},\ q = \frac{A}{D},$$

$$\text{and therefore } M = pA = qB = \frac{AB}{D}.$$

Hence we have this rule: *Multiply the two expressions together and divide the product by the greatest common measure, the quotient will be the least common multiple.*

In practice it is better to divide one of the quantities by the greatest common measure, and multiply the other by the quotient.

If two quantities have no common measure it is plain that their least common multiple is their product.

EXAMPLE. Find the least common multiple of $x^3 - 2x + 1$ and $x^4 - x^3 - x + 1$.

$$
\begin{array}{r|l}
x^3 - 2x + 1 & x^4 - x^3 - x + 1 \;(x - 1 \\
& \underline{x^4 - 2x^2 + x} \\
& - x^3 + 2x^2 - 2x + 1 \\
& \underline{- x^3 \qquad\quad + 2x - 1} \\
& \qquad 2x^2 - 4x + 2
\end{array}
$$

$$
\begin{array}{r|l}
x^2 - 2x + 1 & x^3 - 2x + 1 \;(x + 2 \\
& \underline{x^3 - 2x^2 + x} \\
& 2x^2 - 3x + 1 \\
& \underline{2x^2 - 4x + 2}
\end{array}
$$

$$
\begin{array}{r|l}
x - 1 & x^2 - 2x + 1 \;(x - 1 \\
& \underline{x^2 - x} \\
& - x + 1 \\
& \underline{- x + 1}
\end{array}
$$

Hence $x - 1$ is the greatest common measure.

Again

$$
\begin{array}{r|l}
x - 1 & x^4 - x^3 - x + 1 \;(x^3 - 1 \\
& \underline{x^4 - x^3} \\
& - x + 1 \\
& \underline{- x + 1}
\end{array}
$$

$\therefore\ (x^3 - 1)\,(x^3 - 2x + 1) = x^6 - 2x^4 + 2x - 1$ is the least common multiple required.

32. The least common multiple of three quantities, $A$, $B$ and $C$ is found by determining the least common multiple $D$ of any two of them, as $A$ and $B$, and then finding the least common multiple of $D$ and $C$.

Ex. Find the least common multiple of $x^2 - 1$, $x^3 -$
and $x^2 + 5x + 4$.

The greatest common measure of $x^2 - 1$ and $x^3 - 1$ is easi
found to be $x - 1$; therefore the least common multiple
$\frac{(x^2 - 1)(x^3 - 1)}{x - 1}$, or $(x + 1)(x^3 - 1)$. We must now find th
least common multiple of this quantity and $x^2 + 5x + 4$: it wi
be found that the greatest common measure is $x + 1$; then
fore the least common multiple will be $\frac{(x+1)(x^3-1)(x^2+5x+4}{x+1}$
or $(x^3 - 1)(x^2 + 5x + 4)$, or $x^5 + 5x^4 + 4x^3 - x^2 - 5x - 4$.

## ON FRACTIONS.

33. In arithmetic we define a fraction thus: A fractio
is any part or parts of a unit or whole, and it consists of tw
members, a denominator and a numerator, whereof the forme
shews into how many parts the unit is divided, the latte
shews how many of them are taken in the given case.

Thus $\frac{3}{5}$ denotes that the unit is divided into 5 parts an
that 3 of them are taken; and more generally $\frac{a}{b}$ denotes tha
the unit is divided into $b$ parts and that $a$ of them are take

In algebra we cannot give exactly the same definition, fo
we call any quantity of the form $\frac{a}{b}$ a *fraction*, although $a$ an
$b$ are not necessarily representatives of whole numbers, as the
must be if the fraction be an *arithmetical* or *vulgar* fractio
What is meant by the algebraical fraction $\frac{a}{b}$ is simply thi
that any quantity affected by it is to be *multiplied by* a, an
*divided by* b; this definition however includes that give
above for vulgar fractions, as it ought, because algebra is
more general science than arithmetic, and includes arithmeti
in its rules.

34. *To add two or more fractions together, bring them to*
*common denominator, add the numerators for a new numerator*
*and take the common denominator for a new denominator.*

Let $\frac{a}{b}$, $\frac{c}{d}$ be two fractions,

$$\text{then } \frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd};$$

for in the first place $\frac{a}{b} = \frac{ad}{bd}$, since to multiply and divide by $d$ cannot affect the value of the fraction; and in the next place $\frac{ad}{bd} + \frac{bc}{bd}$ indicates that a quantity is to be divided by $bd$ and multiplied first by $ad$ and then by $bc$, and that these two products are to be added together, whereas the expression $\frac{ad + bc}{bd}$ indicates division by $bd$ and multiplication by $ad + bc$, but this is manifestly the same operation as in the former case, hence $\frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}$.

The rule for subtraction follows at once from that for addition, and is expressed by the formula

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}.$$

35. When any number of fractions are to be reduced to a common denominator, the rule (which requires no proof) is this: Multiply each numerator by every denominator except its own, and all the denominators together for a new denominator; connect the numerators together with their proper signs, and place under them the new denominator; lastly, reduce the fraction so formed to its lowest terms. This rule may be sometimes usefully superseded by the following: Find the least common multiple of the denominators, take this as the new denominator and its product by the several fractions for the several new numerators.

Ex. 1. Reduce the expression $\frac{x^2}{1 - x} - \frac{x^3}{1 - x^2}$ to its most simple equivalent form.

In this case $1 - x^2$ is the least common multiple of the two denominators, and we have

$$\frac{x^2}{1-x} - \frac{x^3}{1-x^2} = \frac{x^2(1+x) - x^3}{1-x^2} = \frac{x^2}{1-x^2},$$

which is the form required.

Ex. 2. Reduce $\dfrac{4}{x+1} - \dfrac{3}{(x+1)^2} + \dfrac{1}{(x+1)^3} - \dfrac{4}{x+2}$ to its simplest equivalent form.

The new denominator will be $(x+1)^3(x+2)$, and the expression may be written thus,

$$\frac{4(x+1)^2(x+2) - 3(x+1)(x+2) + (x+2) - 4(x+1)^3}{(x+1)^3(x+2)},$$

$$\text{which} = \frac{(x+2)\{4(x+1)^2 - 3(x+1) + 1\} - 4(x+1)^3}{(x+1)^3(x+2)}$$

$$= \frac{(x+2)(4x^2+5x+2) - 4(x^3+3x^2+3x+1)}{(x+1)^3(x+2)}$$

$$= \frac{4x^3+13x^2+12x+4-4x^3-12x^2-12x-4}{(x+1)^3(x+2)}$$

$$= \frac{x^2}{(x+1)^3(x+2)},$$

which is the form required.

36. *To multiply two fractions together, multiply the numerators together for a new numerator, and the denominators together for a new denominator.*

Let $\dfrac{a}{b}$, $\dfrac{c}{d}$ be the two fractions, then will $\dfrac{a}{b} \times \dfrac{c}{d} = \dfrac{ac}{bd}$. This is a consequence of the meaning of the symbols, for $\dfrac{a}{b} \times \dfrac{c}{d}$ signifies that the quantity $c$ is to be divided by $d$, then multiplied by $a$, and then divided by $b$; and $\dfrac{ac}{bd}$ signifies that $a$ and $c$ are to be multiplied together and then divided by

the product $bd$; hence the operations indicated in the two cases are the same, the *order* of them is the only difference; but the order of operations has no effect on the result;

$$\therefore \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

The same reasoning will shew that $a \times \frac{c}{d} = \frac{ac}{d}$.

It will be observed here as in other cases that a rather extended sense is given to the term *multiplication;* the student should understand by the term an algebraical operation of which multiplication in arithmetic is the *type.*

37. *To divide one fraction by another, invert the divisor and proceed as in multiplication.*

Since to *divide* is the inverse of to *multiply,* it follows that to divide by $\frac{a}{b}$ is the same thing as to multiply by $\frac{b}{a}$; and hence the rule.

It is hardly necessary to state, that in this and all other cases the fractions resulting from any of the operations for which the rules have been given should be reduced to their lowest terms.

## ON THE THEORY OF DECIMAL FRACTIONS.

38. The principles of algebra which we have been developing are sufficient to enable us to explain and prove the rules of decimal fractions; and as we have been speaking of fractions, this will be a convenient place for introducing the subject.

39. DEF. *A decimal fraction is one which has* 10 *or some power of* 10 *for its denominator.*

Hence a decimal fraction will be represented algebraically by $\frac{N}{10^n}$; where $n$ is the number of decimal places, and $N$ is

the whole number which the decimal would represent if we omitted the decimal point. Thus $1.37 = \frac{137}{10^2}$, in which case $N = 137$ and $n = 2$.

Having obtained this general symbolical representation of a decimal, all the rules will follow with great simplicity.

40. *To prove the rule for the multiplication of decimals.*

Let $\frac{M}{10^m}$, $\frac{N}{10^n}$ be two decimals: then

$$\frac{M}{10^m} \times \frac{N}{10^n} = \frac{MN}{10^{m+n}}.$$

The numerator of this last fraction shews that the decimal quantities are to be multiplied together as if they were whole numbers, the denominator that there must be $m + n$ decimal places, that is, as many as in the multiplier and multiplicand together.

The rule for division may be proved in like manner.

41. *To prove the rule for converting a vulgar fraction into a decimal.*

Let $\frac{A}{B}$ be the vulgar fraction; then

$$\frac{A}{B} = \frac{A \times 10^n \div B}{10^n} \text{ identically.}$$

The numerator shews that we are to add cyphers at pleasure to the right of $A$, (for that is the same thing as multiplying by $10^n$,) and that we are then to divide by $B$; while the denominator indicates that as many places are to be marked off for decimals as we have added cyphers.

42. *To prove that every vulgar fraction must produce either a terminating or a recurring decimal.*

In the division of $A\,10^n$ by $B$ every remainder must be less than $B$, therefore there can be at most only $B - 1$ different remainders; hence if no remainder becomes zero, that is, if the operation does not terminate, a remainder

must recur within $B - 1$ operations at furthest; the figures in the quotient will then recur, and the result will be a recurring decimal.

43. *To determine the form of those vulgar fractions which produce terminating decimals.*

We have as before

$$\frac{A}{B} = \frac{A\,10^n \div B}{10^n}.$$

Now in order that the above may be a terminating decimal, $B$ must divide $A\,10^n$ without remainder; but $B$ cannot divide $A$ since we suppose the fraction to be in its lowest terms, therefore it must divide $10^n$. But it is easy to see that the only numbers which will divide $10^n$ are those which are made up of the factors 2 and 5, because these are the only two numbers which will divide 10. Hence $B$ must have no other factors than 2 and 5, or, speaking algebraically, $B$ must be of the form $2^p 5^q$. For example, $\frac{5}{8}$ will produce a terminating decimal, because $8 = 2^3$; but $\frac{5}{24}$ will not, because 24 is divisible by 3.

These are the most simple propositions relating to decimals; others will be found in Arts. (60) and (107).

## ON INVOLUTION AND EVOLUTION.

44. *Involution* is the process of multiplying a quantity by itself any number of times; *Evolution* is the finding of a quantity which being multiplied into itself a given number of times shall become equal to a given quantity, in other words evolution is the extraction of any *root* of a quantity.

45. The involution of a simple quantity is effected, as we have seen (Art. 25), by multiplying its index, and the evolution by dividing the index; for

$$(a^p)^q = a^{pq} \text{ and } \sqrt[q]{a^p} = a^{\frac{p}{q}}.$$

46. The involution of a polynomial is a very simple though frequently a laborious process, being a process of actual multiplication.

Ex. 1. Find the square of $a + b$.

$$\begin{array}{l} a + b \\ a + b \\ \hline a^2 + ab \\ \qquad ab + b^2 \\ \hline a^2 + 2ab + b^2 = (a + b)^2 \\ \hline \end{array}$$

Ex. 2. Find the cube of $a + b$.

$$\begin{array}{l} a^2 + 2ab + b^2 \\ a + b \\ \hline a^3 + 2a^2b + ab^2 \\ \qquad a^2b + 2ab^2 + b^3 \\ \hline a^3 + 3a^2b + 3ab^2 + b^3 = (a + b)^3 \\ \hline \end{array}$$

It will be easily seen that the labour increases very rapidly with the number of terms in the polynomial and also with the degree of the power.

47. The squaring of a polynomial is rendered very easy by the following theorem:

*The square of any polynomial = the sum of the squares of the terms + twice the product of each two terms.*

To prove this we observe, that in the expression for $(a+b)^2$ the rule obviously holds, for $(a + b)^2 = a^2 + b^2 + 2ab$; now *suppose* the rule to hold for the polynomial $a + b + c + \ldots + l$, so that

$$(a + b + c + \ldots + l)^2 = a^2 + b^2 + c^2 + \ldots + l^2 + 2ab + 2ac + \ldots (1)$$

then introducing another term $m$ we must have

$$(a + b + c \ldots + l + m)^2 = (\overline{a + b + c + \ldots + l} + m)^2$$

$$= (a + b + c \ldots + l)^2 + m^2 + 2(a + b + c + \ldots + l)\, m$$

(because $a+b+c+\ldots+l$ may all be considered as one quantity;)

$$= a^2 + b^2 + c^2 + \ldots + l^2 + 2ab + 2ac + \ldots\ldots\ldots\ldots$$
$$+ m^2 + 2am + 2bm + \ldots \text{ by (1)}$$

= the sum of the squares of the terms + twice the product of each two.

Hence *if* the theorem be true for a polynomial of any number of terms, it will be true for a polynomial of that number of terms increased by one: but the theorem *is* true for an expression of *two* terms, $\therefore$ it is true for one of *three*, $\therefore$ for one of *four*, $\therefore$ &c., $\therefore$ for any polynomial.

The mode of reasoning adopted in the preceding proposition is one of which we shall have to make use again, and is therefore worthy of attentive consideration*.

Examples of the application of this theorem:

$$(x^2 - x + 2)^2 = x^4 + x^2 + 4 - 2x^3 + 4x^2 - 4x$$
$$= x^4 - 2x^3 + 5x^2 - 4x + 4,$$
$$(x^3 - x^2 + x - 1)^2 = x^6 + x^4 + x^2 + 1 - 2x^5 + 2x^4 - 2x^3 - 2x^3 + 2x^2 - 2x$$
$$= x^6 - 2x^5 + 3x^4 - 4x^3 + 3x^2 - 2x + 1.$$

48. *Evolution*, being an inverse process, is not so simple as involution, and the rules for it must be obtained by observing how the power of a compound quantity is formed from the quantity itself.

49. *To extract the square root of a compound quantity.*

Since the square of $a+b$ is $a^2 + 2ab + b^2$, we may obtain a general rule for the extraction of the square root by observing how $a+b$ may be deduced from $a^2 + 2ab + b^2$.

* This is an example of perfect induction. In practical matters our belief is generally founded upon an induction from particular facts, and the greater the number of the facts on which our judgment is founded the greater is the confidence of our belief. But the exactness of mathematical reasoning admits of no conclusion based upon such grounds, because the induction from never so many particular cases cannot do more than establish a strong probability, and the highest probability is altogether different in kind from mathematical certainty. The only kind of induction, which is perfect, and is therefore admissible in mathematics, is when it is shewn, not only that a proposition is true in certain cases, but also that if it be true in one, it is necessarily true in another, and therefore that if true in one it is true in all.

Arrange the expression according to powers (say *descending*) of some letter $a$. The square root of the first term $a^2$ is $a$, which is the first term in the root; subtract its square from the given expression, and bring down the remainder $2ab + b^2$; divide $2ab$ by $2a$, and the result is $b$, the next term in the root; multiply $2a + b$ by $b$ and subtract the product from the remainder $2ab + b^2$. If the operation does not terminate here, *i.e.* if there is another remainder, this will shew that there are more than two terms in the root; in this case we may consider the two terms $a + b$ already found as one, and as corresponding to the term $a$ in the preceding operation; and the square of this quantity, viz. $a^2 + 2ab + b^2$, having been by the preceding process subtracted from the given expression, we may divide the remainder by $2(a + b)$ for the next term in the root, and for a new subtrahend multiply $2(a + b)$ + the new term by that new term. The process may be repeated as often as necessary.

$$
\begin{array}{rl}
 & a^2 + 2ab + b^2 \ (a + b \\
 & a^2 \\
\hline
2a + b) & + 2ab + b^2 \\
 & 2ab + b^2 \\
\hline
\end{array}
$$

EXAMPLE. Find the square root of

$$
\begin{array}{rl}
 & x^6 - 2x^5 + 2x^3 + x^2 - 2x + 1 \quad (x^3 - x + 1 \\
 & x^6 \\
\hline
2x^3 - x) & - 2x^4 + 2x^3 + x^2 \\
 & - 2x^4 + x^2 \\
\hline
2x^3 - 2x + 1) & 2x^3 - 2x + 1 \\
 & 2x^3 - 2x + 1 \\
\hline
\end{array}
$$

50. *To extract the cube root of a compound quantity.*

We have seen (Art. 46) that $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. Hence we deduce the rule as in the case of the square root. Arrange the expression according to descending powers of $a$, the cube root of the first term $a^3$ is $a$, the first term of the root; subtract its

$$
\begin{array}{rl}
 & a^3 + 3a^2b + 3ab^2 + b^3 \ (a + b \\
 & a^3 \\
\hline
3a^2) & 3a^2b + 3ab^2 + b^3 \\
 & 3a^2b + 3ab^2 + b^3 \\
\hline
\end{array}
$$

cube from the given expression, and bring down the remainder; divide the first term by $3a^2$, the quotient is $b$ the second term of the root; subtract the quantity $3a^2b + 3ab^2 + b^3$; if there is no remainder the root is extracted; if there is, we must proceed as before, considering $a + b$ as one term, corresponding to $a$ in the first operation.

EXAMPLE. Find the cube root of

$$
\begin{array}{lllll}
 & x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64 \;(x^2 + 4x + 4 & & & \\
 & x^6 & & & \\
\hline
3x^4) & 12x^5 + 60x^4 + 160x^3 & & & \\
(3a^2b =) & 12x^5 & & & \\
(3ab^2 =) & \qquad + 48x^4 & & & \\
(b^3 =) & \qquad\qquad + 64x^3 & & & \\
\hline
3x^4 + 24x^3 + 48x^2) & 12x^4 + 96x^3 + 240x^2 + 192x + 64 & & & \\
(3a^2b =) & 12x^4 + 96x^3 + 192x^2 & & & \\
(3ab^2 =) & \qquad\qquad 48x^2 + 192x & & & \\
(b^3 =) & \qquad\qquad\qquad + 64 & & & \\
\hline
\end{array}
$$

The cases of the *square* and *cube* root have been given separately on account of their more frequent occurrence, and in order to explain the rules for their extraction in arithmetic, but they are both included in the following investigation of the method of extracting the $n^{\text{th}}$ root of a polynomial.

51. We must premise the following

LEMMA. *The first two terms of* $(a + b)^n$, *where* n *is a positive whole number, are* $a^n + na^{n-1}b$.

For by actual multiplication this is seen to be the case when $n = 2$, because $(a + b)^2 = a^2 + 2ab + b^2$, and when $n = 2$ $a^n + na^{n-1}b = a^2 + 2ab$. Now, *suppose* the proposition to be true for any value of $n$, i. e. *suppose* that

$$(a + b)^n = a^n + na^{n-1}b + \text{terms involving lower powers of } a,$$

$$
\begin{aligned}
\text{then } (a + b)^{n+1} &= (a + b)\{a^n + na^{n-1}b + \dots\dots\dots\} \\
&= a^{n+1} + na^n b + \dots\dots\dots \\
&\qquad + a^n b + \dots\dots\dots
\end{aligned}
$$

by actual multiplication,

$$= a^{n+1} + (n+1)a^n b + \dots\dots\dots$$

which shews that *if* the Lemma be true for any value of $n$, it is true for the whole number next greater; but it *is* true when $n = 2$, $\therefore$ it is true when $n = 3$, $\therefore$ when $n = 4$, $\therefore$ &c., $\therefore$ generally true.

This is another example of the mode of proof used in Art. 47.

52. *To find the* n<sup></sup>th *root of a polynomial.*

Suppose the $n^{\text{th}}$ root to be $a + bx + cx^2 + \dots\dots\dots$ which is arranged according to ascending powers of some letter $x$: then the given polynomial is $(a + bx + cx^2 + \dots\dots\dots)^n$, which however we are to suppose expanded and arranged according to ascending powers of $x$.

Now by the Lemma

$$(a + \overline{bx + cx^2 + \dots\dots})^n = a^n + na^{n-1}(bx + cx^2 + \dots\dots) + \dots\dots$$
$$= a^n + na^{n-1}bx + \text{terms involving powers of } x \text{ above the } \textit{first}.$$

The *first* term $a$ of the required root is known by inspection, being the $n^{\text{th}}$ root of $a^n$; subtract $a^n$ from the given expression, then the first term of the remainder is $na^{n-1}bx$; divide this by $na^{n-1}$ and we have $bx$ the *second* term of the root.

Again, by the Lemma,

$$(\overline{a + bx} + \overline{cx^2 + \dots\dots})^n = (a + bx)^n + n(a + bx)^{n-1}(cx^2 + \dots\dots) + \&c.$$
$$= (a + bx)^n + na^{n-1}cx^2 + \text{terms involving powers of } x \text{ above the } \textit{second}.$$

The terms $a$ and $bx$ are already known; if then we subtract $(a + bx)^n$ from the given expression, the first term of the remainder will be $na^{n-1}cx^2$, dividing which by the same quantity as in the first process, viz. $na^{n-1}$, we have $cx^2$ the *third* term of the root.

By precisely similar reasoning it will appear, that if we subtract $(a + bx + cx^2)^n$ from the given polynomial and divide

the first term of the remainder by $na^{n-1}$, we shall obtain the *fourth* term of the root; and so on.

EXAMPLE. Extract the fourth root of

$$
\begin{array}{l}
x^8 - 8x^7 + 28x^6 - 56x^5 + 70x^4 - 56x^3 + 28x^2 - 8x + 1 \;\left(x^2 - 2x + 1\right. \\
x^8 \\
\hline
4x^6) - 8x^7 \\
\hline
x^8 - 8x^7 + 24x^6 - 32x^5 + 16x^4 \left\{= (x^2 - 2x)^4\right\} \\
\hline
\qquad 4x^6)\; 4x^6 - \&c. \\
\hline
x^8 - 8x^7 + 28x^6 - 56x^5 + 70x^4 - 56x^3 + 28x^2 - 8x + 1 \left\{= (x^2 - 2x + 1)^4\right\}. \\
\hline
\end{array}
$$

53. The preceding investigations are quite necessary in order to understand fully the theory of the extraction of the square and cube roots of *numbers*.

54. *On the rule of pointing in the extraction of the square root of a number.*

Every number consisting of one figure or digit is less than 10, and therefore the square of a number of one figure is less than $10^2$; more generally, every number of $n$ figures is less than $10^n$, (because $10^n$ represents 1 followed by $n$ cyphers), and therefore the square of such a number is less than $10^{2n}$, but also every number of $n$ figures is not less than $10^{n-1}$, and therefore its square is not less than $10^{2n-2}$; now $10^{2n-2}$ is the smallest number of $2n - 1$ figures, and $10^{2n}$ the smallest of $2n + 1$ figures, consequently the square of a number of $n$ figures has either $2n$ or $2n - 1$ figures. This being the case if we put a point over the unit's place of a number of which the root is to be extracted, and point every second figure from right to left, the number of points will always be equal to the number of figures in the root: if the number of figures be even, the number will be divided into compartments of two each; if odd, the last compartment will contain only a single figure.

Ex. $1\dot{7}2\dot{4}3\dot{6}$, $\dot{2}1\dot{5}4\dot{7}$: each of these numbers has three figures in its square root.

The rule for extracting the square root of a number is an adaptation of that for extracting the square root of an algebraical expression. The nature of the adaptation will be seen best by an example.

Let it be required to extract the square root of 2116.

Point the unit's place and every second figure; find the greatest number the square of which is not greater than the number expressed by the first period; in the example 21 is the first period and $4^2$ is not greater than 21, hence 4 is the first figure in the root. Then subtract the square of the number thus found from the first period and bring down the second; divide this number, omitting the last figure, by twice the number already found, the quotient is the second figure of the root; in the example we divide 51 by 8, which gives 6 for the second figure. Annex the figure thus found to the divisor, and multiply the divisor so increased by the figure of the root last found, to form the subtrahend; in the example 86 is multiplied by 6, which gives the subtrahend 516. If there be more periods to be brought down, the operation must be repeated.

$$
\begin{array}{rl}
 & 2\dot{1}1\dot{6}\ (46 \\
 & 16 \\
\hline
86) & 516 \\
 & 516 \\
\hline
\end{array}
$$

55. *On pointing in the extraction of the cube root.*

It may be shewn in the same way as in the case of the square root (Art. 54), that the cube of a number of $n$ figures contains $3n$, $3n-1$, or $3n-2$ figures, and therefore that if we put a point over the unit's place and on each third figure we shall have as many periods of figures as there are figures in the root.

56. The rule for the extraction of the cube root of a number is deduced from that for the extraction of the cube root of an algebraical expression, in the same way as in the case of the square root.

Let it be required to extract the cube root of 12167.

Point the number according to the rule; in the example there are two periods. Find the greatest number the cube of which is not greater than the number expressed by the first period, this will be the first figure in the root; in the example it is 2: in order to compare this operation with the algebraical one, call this figure with a cypher affixed to it $a$, so that in the example $a = 20$, and let $b$ be the next figure required. Subtract the cube of the figure already found from the first period and bring down the second; divide this by $3a^2$, and the quotient will *probably* be the next figure $b$ of the root; in the example we find $b = 3$. Form the subtrahend by compounding the formula $3a^2b + 3ab^2 + b^3$, where $a$ and $b$ have the values already found; if this subtrahend is too large, take the number next less than that found for $b$ and try again, and so on until you find a subtrahend sufficiently small. If there are more than two figures in the root, bring down the next period and proceed as before; the subtrahend will always be found by the formula $3a^2b + 3ab^2 + b^3$, it being remembered that $a$ stands for all the figures already found with a cypher affixed; suppose for example, that in any extraction the figures 274 had been found and that 6 was the next, then $a = 2740$ and $b = 6$.

$$
\begin{array}{rl}
 & \dot{1}21\dot{6}7\ (\overset{a}{20} + \overset{b}{3} = 23 \\
 & 8 \\
\cline{2-2}
3a^2 = 1200) & 4167 \\
 & 3600 = 3a^2b \\
 & 540 = 3ab^2 \\
 & 27 = b^3 \\
\cline{2-2}
 & 4167 = \text{subtrahend.} \\
\cline{2-2}
\end{array}
$$

57. *To explain why it is that the rule given for finding the successive figures of the cube root of a number frequently gives a number too large.*

Suppose the part of the root found to be $a$, and the next digit $b$, then the rule is to subtract $a^3$ and divide by $3a^2$; but the actual quantity given by this rule is

$$\frac{3a^2b + 3ab^2 + b^3}{3a^2} = b + \frac{3ab^2 + b^3}{3a^2};$$

consequently the result given by the rule differs from the required number $b$ by the quantity $\frac{3ab^2+b^3}{3a^2}$, which may very well be greater than 1, and if so the number given by the rule will be too great. The rule is more likely to be in error at the commencement of the operation, because then $a$ is not so great as afterwards.

The name of *trial divisor* has been very properly assigned to $3a^2$.

58. Hence we see why in arithmetic no rule can be given for the extraction of the higher roots; for the rule for the cube root becomes, as we have seen, uncertain; and if in the case of high roots we adopted the method of Art. 52, we should find that the *trial divisor* $na^{n-1}$ would scarcely ever give us any help in discovering the figures of the root.

59. The distinction between the algebraical and arithmetical operations will be seen at once by observing the difference in the operations of squaring (or raising to any power) an algebraical expression and a number. We have

$$(ax+b)^2 = a^2x^2 + 2abx + b^2.$$

Now suppose $x = 10$, then $10a + b$ represents a number having digits $a$ and $b$; but $(10a+b)^2$ is not represented arithmetically by $10^2a^2 + 10.2ab + b^2$, *unless* $a^2$, $2ab$, *and* $b^2$ *be each less than* 10; and thus the square of a number may have lost algebraically the type of the number itself.

For example, $18 = 10 + 8$, but $18^2$ is not represented by $10^2 + 10.16 + 64$, but by $10^2.3 + 10.2 + 4$.

60. *To investigate a rule for pointing in the extraction of the square root of a decimal quantity.*

A decimal quantity may be represented by $\frac{N}{10^n}$, where $N$ represents the number supposing the decimal point omitted, and $n$ is the number of decimal places. Now $n$ is either odd or even; if it is odd, multiply numerator and denominator by 10, and let the quantity thus modified be represented by $\frac{M}{10^{2m}}$.

Then $\sqrt{\dfrac{M}{10^{2m}}} = \dfrac{\sqrt{M}}{10^m}$, a formula which indicates that the square root of $M$ is to be extracted as in whole numbers, and that $m$ places are to be marked off for decimals; but in pointing $M$ it is to be observed that, since we made the number of decimal places even, a point will necessarily fall on the original unit's place. Hence we have this rule: *Put a point over the unit's place, and point every second figure right and left.*

The rule for pointing in the extraction of the cube root may be found in a similar manner.

61. *When* p *figures of a square root have been obtained by the ordinary method,* p − 1 *more may be obtained by division only.*

Let $a$ be the part of the root already obtained, $x$ the part consisting of $p-1$ figures which we wish to obtain, and let $N$ be the whole root, so that

$$N = a\,10^{p-1} + x,$$

$$\therefore\ N^2 = a^2 10^{2p-2} + 2ax10^{p-1} + x^2.$$

Subtracting $a^2 10^{2p-2}$ from each side of this equation and dividing by $2a10^{p-1}$, we have

$$\frac{N^2 - a^2 10^{2p-2}}{2a10^{p-1}} = x + \frac{x^2}{2a10^{p-1}}.$$

From this it appears that the division above indicated will give us $x$ correctly, if we can prove $\dfrac{x^2}{2a10^{p-1}}$ to be a proper fraction.

Now $x$ consists of $p-1$ figures, and $\therefore$ is $< 10^{p-1}$;

$$\therefore\ x^2 \text{ is } < 10^{2p-2};$$

but $a$ consists of $p$ figures, and $\therefore$ is not $<10^{p-1}$;

$$\therefore\ \frac{x^2}{a10^{p-1}} < \frac{10^{2p-2}}{10^{2p-2}} < 1;$$

$$\therefore\ \frac{x^2}{2a10^{p-1}} < \frac{1}{2},$$

and hence the division will give us the $p-1$ figures of $x$ correctly.

## ON EQUATIONS.

62. An *equation* has already been defined to be an algebraical sentence expressing the equality of two algebraical expressions, or (which is the same thing) of an algebraical expression to zero.

If an unknown quantity is involved, the equation involving it serves to determine the unknown quantity, and it is our business now to lay down rules for the performance of this process, which is called the *solution* of the equation.

If when cleared of radicals an equation involves only the *first* power of the unknown quantity $x$, it is called a *simple* equation; if it involves $x^2$ also, it is called a *quadratic;* and, generally, if it involves $x^n$ it is said to be *an equation of* n *dimensions, or of the* n<sup>th</sup> *degree.* We shall in this treatise be concerned only with *simple* and *quadratic* equations.

A value of $x$ which satisfies an equation is called a *root* of the equation.

63. Suppose we have the equation $x + a = b$; then subtracting $a$ from each side we have $x = b - a$, that is, a quantity may be placed on the other side of an equation if its sign be changed. This process, which is one of the most frequent in the solution of equations, is called *transposition.*

64. It is manifest that if the same operation be performed on the two sides of an equation, the equality will still subsist; we may therefore multiply or divide the two sides of an equation by the same quantity, or may raise the two sides to the same power, or extract any root of both sides. If, for example,

$$x^2 + 2x - 3 = 3x^2 + 1,$$

then will the following equations hold good,

$$P(x^2 + 2x - 3) = P(3x^2 + 1),$$

$$\frac{x^2 + 2x - 3}{P} = \frac{3x^2 + 1}{P},$$

$$(x^2 + 2x - 3)^n = (3x^2 + 1)^n,$$

$$\sqrt[n]{x^2 + 2x - 3} = \sqrt[n]{3x^2 + 1}.$$

65. A frequent process in the solution of equations is *clearing the equation of radicals;* this is done by putting any radical of which we desire to rid the equation on one side by itself, and transposing all the other terms to the other side, we then raise both sides of the equation to the power indicated by the radical, which consequently disappears. The process will be understood best by an example: suppose

$$\sqrt{x+1} + \sqrt{x-1} = 2.$$

The process of clearing this equation of radicals will stand as follows:

| | |
|---|---|
| transposing, | $\sqrt{x+1} = 2 - \sqrt{x-1}$, |
| squaring, | $x + 1 = 4 - 4\sqrt{x-1} + x - 1$, |
| transposing, | $4\sqrt{x-1} = 2$, |
| dividing by 4, | $\sqrt{x-1} = \frac{1}{2}$, |
| squaring, | $x - 1 = \frac{1}{4}$; |

which is a simple equation free from radicals.

It is obvious that it is generally impossible to ascertain the *degree* of an equation, that is, whether it is simple, quadratic, or of any higher degree, until it has been cleared of radicals.

66. An equation is cleared of fractions by multiplying both sides of it by the least common multiple of the denominators. It is however perhaps practically the easiest method to multiply by each of the denominators in succession, and make such simplifications as the case allows after each multiplication.

## ON SIMPLE EQUATIONS.

67. *To find the value of an unknown quantity in a simple equation.*

An equation given as a simple equation may involve radicals; if so, let them be got rid of first. Next, clear the equation of fractions. Next, transpose all terms involving the unknown quantity to the left-hand side, and all terms

involving only known quantities to the right-hand side of the equation. Divide both sides by the coefficient, or sum of the coefficients, of the unknown quantity, and the value required is obtained.

It is manifest that a simple equation can have only one solution.

Ex. 1.
$$2x + 3 = 3x - 1,$$
$$3x - 2x = 3 + 1,$$
$$x = 4.$$

Ex. 2.
$$\frac{2x-1}{3} + \frac{3x-2}{2} = \frac{5}{6},$$
$$2(2x-1) + 3(3x-2) = 5,$$
$$4x + 9x = 2 + 6 + 5,$$
$$13x = 13,$$
$$x = 1.$$

Ex. 3.
$$\sqrt{x+1} - \sqrt{x-1} = 2,$$
$$\sqrt{x+1} - 2 = \sqrt{x-1},$$
$$x + 1 + 4 - 4\sqrt{x+1} = x - 1,$$
$$4\sqrt{x+1} = 6,$$
$$\sqrt{x+1} = \frac{3}{2},$$
$$x + 1 = \frac{9}{4},$$
$$x = \frac{9}{4} - 1 = \frac{5}{4}.$$

It may be remarked that we should have obtained the same result as in this last example, if we had proceeded in the same manner with the equation

$$\sqrt{x+1} + \sqrt{x-1} = 2,$$

and in fact the value $\frac{5}{4}$ will verify this equation but not the equation

$$+\sqrt{x+1} - \sqrt{x-1} = 2.$$

The explanation of this apparent difficulty is, that whenever a quantity of the form $\sqrt{a}$ is given, it must be supposed to have the sign $\pm$ attached to it, because $(+\sqrt{a})^2$ and $(-\sqrt{a})^2$ are each of them equal to $a$; so that the equation of Ex. 3 might more properly have been written

$$\pm\sqrt{x+1} \pm \sqrt{x-1} = 2,$$

and when by expelling the radicals we have obtained a numerical result from such an equation as this, it will generally be necessary to choose particular signs for the radicals in order that the substitution of the value obtained may arithmetically satisfy the equation.

## ON QUADRATIC EQUATIONS.

68. All the rules for the simplification of equations which have already been given apply equally to quadratic equations; indeed, as has been observed, it is not always, until the simplification has been effected, that we are able to say whether the equation is simple or quadratic. By such simplification the equation, if a quadratic, will be reduced to one of these forms*,

$$x^2 = b,$$
$$\text{or } x^2 + ax = b.$$

In the *first* case we have at once, by extracting the square root of both sides,

$$x = \pm\sqrt{b}.$$

Let the student take particular notice of the double sign prefixed to the radical, and *which in the solution of quadratic equations ought always to be prefixed,* because $-\sqrt{b}$ and $+\sqrt{b}$ satisfy the equation $x^2 = b$ equally well, since the square of either of them is $+b$.

In the *second* case the solution is not so obvious; but it is easily effected, by observing that the quantity $\frac{a^2}{4}$ added to

* These two forms are sometimes distinguished as *Pure* Quadratics and *Adfected* Quadratics.

each side of the equation will make the left-hand side a complete square; in fact, we have

$$x^2 + ax + \frac{a^2}{4} = b + \frac{a^2}{4},$$

which may be written thus

$$\left(x + \frac{a}{2}\right)^2 = b + \frac{a^2}{4},$$

extracting the square root of each side we have

$$x + \frac{a}{2} = \pm \sqrt{b + \frac{a^2}{4}},$$

transposing, $$x = -\frac{a}{2} \pm \sqrt{b + \frac{a^2}{4}}.$$

The two values of $x$,

$$-\frac{a}{2} + \sqrt{b + \frac{a^2}{4}} \text{ and } -\frac{a}{2} - \sqrt{b + \frac{a^2}{4}},$$

satisfy the equation $x^2 + ax = b$, and are called its *roots*. The student may, if he pleases, actually substitute in the equation either of the expressions which we have found, and he will find the result to be an identity.

Hence, further, we see that every quadratic has *two roots* and no more*, and in solving quadratics the student should be careful always to represent both roots; and it may be mentioned that every equation has as many roots as it has dimensions, that is, a *cubic* has *three* roots, a *biquadratic four*, and so on.

* It may be shewn by a direct process that a quadratic has only two roots; for if possible, suppose that $\alpha$, $\beta$, $\gamma$ are roots of the equation

$$x^2 + ax + b = 0;$$
$$\therefore \alpha^2 + a\alpha + b = 0 \quad (1),$$
$$\beta^2 + a\beta + b = 0 \quad (2),$$
$$\gamma^2 + a\gamma + b = 0 \quad (3).$$

Subtracting (2) from (1) we have

$$\alpha^2 - \beta^2 + a(\alpha - \beta) = 0,$$
$$\text{or } \alpha + \beta + a = 0.$$

In like manner, from subtracting (3) from (1) there results,

$$\alpha + \gamma + a = 0;$$

$\therefore \beta = \gamma$, which is not true. Therefore a quadratic cannot have more than two roots.

69. The following form of a quadratic is included in the preceding, by supposing $b$ negative, but it is worthy of separate consideration. Suppose

$$x^2 + ax = -b.$$

Completing the square as before,

$$x^2 + ax + \frac{a^2}{4} = \frac{a^2}{4} - b,$$

$$\therefore \quad x + \frac{a}{2} = \pm\sqrt{\frac{a^2}{4} - b}.$$

$$\text{and } x = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b}.$$

Now, suppose that $\frac{a^2}{4}$ is less than $b$, then $\frac{a^2}{4} - b$ is a negative quantity, and the expression $\sqrt{\frac{a^2}{4} - b}$ represents an operation *which cannot be performed*, for there is no quantity the square of which is negative. Quantities of this kind are called *impossible* or *imaginary*, and the roots of an equation when they involve such quantities are called *impossible* or *imaginary* roots; so far as we are concerned at present however, roots of this kind are quite as much the object of our search as *real* roots, since the question is merely this, What symbolical quantity substituted for $x$ and operated upon according to the rules of algebra will satisfy a given equation * ?

* When the solution of a problem leads to a quadratic equation, the roots of which are imaginary, it may generally be concluded that the problem is impossible, that is, that the conditions supposed in its enunciation to be satisfied are inconsistent. As for instance, suppose it were required to divide the number 10 into two parts such that their product should be 30. If we call one part $x$ and the other $10-x$, the equation of the problem will be

$$x(10-x) = 30,$$

$$\text{or } x^2 - 10x = -30,$$

$$x^2 - 10x + 25 = -5,$$

$$x = 5 \pm \sqrt{-5},$$

imaginary values, because the problem is an impossible problem. It may be observed, that important meaning may in many cases be assigned to these imaginary quantities; but to enter into the discussion of that subject would be foreign to the elementary character of this work.

One kind of problem however may be mentioned in this place as being of practical utility. Suppose instead of the above problem it were proposed to divide a number $a$

It is manifest that if one root of a quadratic is imaginary the other is also imaginary.

70. *If* $\alpha$ *and* $\beta$ *be the two roots of a quadratic equation*

$$x^2 + ax + b = 0,$$

*then will*

$$x^2 + ax + b = (x - \alpha)(x - \beta).$$

This may be proved by substituting for $\alpha$ and $\beta$ the values already obtained for the roots of the equation; it may also be proved (more elegantly) thus.

Since $\alpha$ and $\beta$ are roots of the equation, we have

$$\alpha^2 + a\alpha + b = 0,$$

$$\beta^2 + a\beta + b = 0;$$

$$\therefore \quad \alpha^2 - \beta^2 + a(\alpha - \beta) = 0,$$

$$\text{or (dividing by } \alpha - \beta) \; a = -(\alpha + \beta)$$

$$\therefore \quad b = -\alpha^2 + (\alpha + \beta)\alpha = \alpha\beta.$$

Hence

$$x^2 + ax + b = x^2 - (\alpha + \beta)x + \alpha\beta,$$

$$= (x - \alpha)(x - \beta).$$

If then we can see by inspection of an equation a quantity of the form $x - \alpha$, which will divide the equation without remainder, we shall know one of the roots of the equation, and dividing by $x - \alpha$, we shall have the factor $x - \beta$ left, which will give us the other root.

It may be mentioned, that in equations of all degrees, if $\alpha$ be a root of the equation, $x - \alpha$ will divide without re-

into two parts, the product of which should be the *greatest possible*. If we call one part $x$, the other $a - x$, and the product $y$, we have this equation,

$$x(a-x) = y,$$

$$\text{or } x^2 - ax = -y,$$

$$x^2 - ax + \frac{a^2}{4} = \frac{a^2}{4} - y,$$

$$x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - y}.$$

Now if $y$ be greater than $\frac{a^2}{4}$, $x$ becomes imaginary, or the problem impossible; therefore the greatest value $y$ can have is $\frac{a^2}{4}$, and then $x = \frac{a}{2}$. Hence the product of the two parts of a number is greatest when the number is divided into two *equal* parts, a result which may be easily verified geometrically. A variety of problems involving questions of *maximum* or *minimum* may be solved in like manner.

mainder*. Hence an equation which appears as a cubic may sometimes be readily reduced to a quadratic, by expelling a root which is visible on inspection. For example, suppose we have the equation,

$$x^3 - 1 = 6x^2 - 6,$$

$$\text{or } x^3 - 6x^2 + 5 = 0.$$

Here it is manifest that 1 is a root, therefore $x - 1$ will divide without remainder, and if the division be performed we shall have only a quadratic to solve.

71. An equation may sometimes be treated as a quadratic which has higher powers of $x$ in it than the second; in fact every equation is virtually a quadratic which has only two powers of $x$ involved, one of which is twice as high as the other. See Ex. 4.

72. An equation which presents itself under the form of a biquadratic may sometimes be solved as a quadratic. For example, let us take the equation

$$x^4 - 2x^2 - 400x = 999.$$

Add the quantity $4x^2 + 400x + 1$ to each side, and the equation becomes

$$x^4 + 2x^2 + 1 = 4x^2 + 400x + 1000$$

$$\text{or } x^2 + 1 = \pm (2x + 100),$$

and by the solution of these quadratics the roots of the original equation may all be found†.

* The proof is very simple. Let $P = 0$ be any equation of which $a$ is a root, that is, let $P$ be an expression of the form $x^n + ax^{n-1} + bx^{n-2} + \&c.$ such that when $a$ is written for $x$ the whole quantity becomes zero. Also let $P$ be divided by $x - a$ until we come to a remainder not involving $x$, and let the quotient be $Q$ and the remainder $R$; so that

$$P = Q(x - a) + R.$$

Now this is not a mere equation, but an *identity*; that is to say, it only represents $P$ in a different form, and it will therefore be true whatever value we choose to give to $x$. Let then $x = a$; then by hypothesis $P$ becomes 0, and therefore also $R = 0$, in other words $x - a$ will divide $P$ without remainder.

† It may be worth while to remark that equations which can be solved after the manner of the preceding may be constructed by assigning values to the quantities $a$, $b$, $c$ in the following formula,

$$(x^2 + a)^2 = b(x + c)^2,$$

which assumes, when arranged according to powers of $x$, the form

$$x^4 + (2a - b)x^2 - 2bcx + a^2 - bc^2 = 0.$$

Ex. 1.
$$2x^2 + 3x + 1 = x^2 + 5,$$
$$x^2 + 3x = 4,$$
$$x^2 + 3x + \frac{9}{4} = \frac{9}{4} + 4 = \frac{25}{4},$$
$$x + \frac{3}{2} = \pm\frac{5}{2},$$
$$x = \frac{-3 \pm 5}{2} = 1 \text{ or } -4.$$

Ex. 2.
$$\frac{2}{x-1} + \frac{1}{x} = 4,$$
$$2x + x - 1 = 4x(x-1),$$
$$4x^2 - 4x - 3x = -1,$$
$$4x^2 - 7x = -1,$$
$$x^2 - \frac{7x}{4} = -\frac{1}{4},$$
$$x^2 - \frac{7x}{4} + \frac{49}{64} = \frac{49}{64} - \frac{16}{64} = \frac{33}{64},$$
$$x = \frac{7 \pm \sqrt{33}}{8}.$$

Ex. 3.
$$x^2 + 2x + 3 = 4x + 1,$$
$$x^2 - 2x = -2,$$
$$x^2 - 2x + 1 = -1,$$
$$x - 1 = \pm\sqrt{-1},$$
$$x = 1 \pm \sqrt{-1}.$$

Ex. 4.
$$x^6 + 2x^3 = 8,$$
$$x^6 + 2x^3 + 1 = 9,$$
$$x^3 + 1 = \pm 3,$$
$$x^3 = -1 \pm 3 = 2 \text{ or } -4,$$
$$x = \sqrt[3]{2} \text{ or } -\sqrt[3]{4}.$$

A particular case is that in which $b = 2a$; and the form then becomes
$$x^4 - 4acx + a^2 - 2ac^2 = 0;$$
by giving to $a$ and $c$ different values we may construct an infinite variety of equations the method of solving which might not be at first sight obvious.

For example, let $c = a = 1$, and we have the equation
$$x^4 - 4x = 1.$$

## ON SIMULTANEOUS EQUATIONS.

73. We have seen how it is possible to find the value of $x$ which satisfies a given simple or quadratic equation; but sometimes the problem is presented of finding *two unknown quantities from two equations;* two equations which are thus given to determine two quantities $x$ and $y$, involved in both, are said to be *simultaneous*.

74. The simple rule for the solution of such equations is to find the value of one of the unknown quantities ($y$), in terms of the other ($x$) from one equation, and substitute the value, so found, in the other; we shall thus have an equation involving $x$ only for determining $x$, and this may be a simple equation or a quadratic, according to circumstances.

The process just described is not always in practice the most convenient; it is manifest that it does not signify in what manner the quantity $y$ is got rid of between the two equations, and we may therefore give this rule; *Eliminate*, (i.e. get rid of) $y$ between the two equations, and obtain $x$ from the result. The ingenuity of the student will frequently be exercised in determining the most convenient mode of elimination.

75. It is not difficult to see that *two* equations, and no more, are necessary for the determination of *two* unknown quantities; in like manner, three equations will be necessary and sufficient to determine three unknown quantities; and so on. The name *simultaneous* is applied to any such system of equations, however many there may be.

Ex. 1. Given
$$\left.\begin{array}{ll} ax + by = c & (1) \\ a'x + b'y = c' & (2) \end{array}\right\}$$
to find $x$ and $y$.

Multiply (1) by $b'$, and (2) by $b$, and the equations become
$$ab'x + bb'y = b'c,$$
$$a'bx + bb'y = bc'.$$

Subtract one of these from the other, and there results

$$(ab' - a'b)\,x = b'c - bc';$$

$$\therefore\ x = \frac{b'c - bc'}{ab' - a'b}.$$

To find $y$ we have from (1)

$$y = \frac{1}{b}(c - ax),$$

writing for $x$ its value,

$$y = \frac{1}{b}\left(c - a\frac{b'c - bc'}{ab' - a'b}\right)$$

$$= \frac{1}{b}\,\frac{abc' - a'bc}{ab' - a'b} = \frac{ac' - a'c}{ab' - a'b}.$$

Thus we have found both $x$ and $y$; the latter however might have been determined more neatly by treating (1) and (2) as we did in finding $x$, that is, multiplying the former by $a'$, the latter by $a$, and subtracting the resulting equations.

76. It may perhaps be also worth while to remark, even in this early example, that, in a system of equations such as (1) and (2), when $x$ has been found, $y$ may be known by inspection. For it is to be observed, that in (1) and (2) $x$ bears the same relation to $a$ and $a'$, that $y$ bears to $b$ and $b'$; in fact, if in those equations we write $b$ for $a$, and $b'$ for $a'$, and lastly, interchange $x$ and $y$, the equations remain unchanged; hence we conclude, that the value of $y$ may be obtained from that of $x$ by writing $b$ for $a$, $b'$ for $a'$, and of course $a$ for $b$, and $a'$ for $b'$.

Now $$x = \frac{b'c - bc'}{ab' - a'b};$$

$$\therefore\ y = \frac{a'c - ac'}{ba' - b'a},$$

which is the same result as before, though written in a manner slightly different.

Ex. 2.
$$\left.\begin{aligned} \frac{x-1}{2}+\frac{2y-1}{3}&=1 \qquad (1)\\ 2x+1-\frac{y+2}{3}&=0 \qquad (2)\end{aligned}\right\}.$$

These equations must first be cleared of fractions and put in their simplest form.

Multiplying (1) by 6, there results

$$3x-3+4y-2=6,$$
$$\text{or } 3x+4y=11 \qquad (3).$$

Multiplying (2) by 3,

$$6x+3-y-2=0,$$
$$\text{or } 6x-y+1=0;$$
$$\therefore\ y=6x+1 \qquad (4).$$

Writing this value for $y$ in (3),

$$3x+4(6x+1)=11,$$
$$27x=11-4=7;$$
$$\therefore\ x=\frac{7}{27};$$

and therefore from (4)

$$y=6\times\frac{7}{27}+1=\frac{14}{9}+1=\frac{23}{9}.$$

Ex. 3.
$$\left.\begin{aligned} x+y&=a \qquad (1)\\ xy&=b \qquad (2)\end{aligned}\right\}.$$

From (1) $\qquad y=a-x,$

putting this value for $y$ in (2), we have

$$x(a-x)=b,$$
$$\text{or } x^2-ax=-b.$$

Completing the square according to the rule of Art. 68,

$$x^2-ax+\frac{a^2}{4}=\frac{a^2}{4}-b,$$

$$x - \frac{a}{2} = \pm\sqrt{\frac{a^2}{4} - b},$$

$$x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b},$$

$$\text{and } y = a - x = \frac{a}{2} \mp \sqrt{\frac{a^2}{4} - b}.$$

**Ex. 4.**

$$x + y + z = 0 \qquad (1),$$
$$x + 2y + 3z = 1 \qquad (2),$$
$$2x + y + 3z = 2 \qquad (3).$$

Subtracting (1) from (2),

$$y + 2z = 1 \qquad (4).$$

Multiplying (1) by 2, and subtracting the result from (3),

$$-y + z = 2 \qquad (5).$$

Adding (4) and (5),

$$3z = 3,$$
$$z = 1;$$

therefore from (4) $\quad y = 1 - 2z = 1 - 2 = -1,$

and from (1) $\quad x = -y - z = 1 - 1 = 0.$

**Ex. 5.**

$$x + y + z = 1 \qquad (1),$$
$$(b + c)\,x + (c + a)\,y + (a + b)\,z = 0 \qquad (2),$$
$$bcx + cay + abz = 0 \qquad (3).$$

Multiply (2) by $bc$ and (3) by $b + c$ and subtract, then

$$\{(a + c)\,bc - (b + c)\,ac\}\,y + \{(a + b)\,bc - (b + c)\,ab\}\,z = 0,$$

$$\text{or } (bc^2 - ac^2)\,y + (b^2c - ab^2)\,z = 0;$$

$$\therefore\ y = -\frac{b^2}{c^2}\frac{c - a}{b - a}\,z.$$

Similarly we should find (see the remarks on Example 1),

$$x = -\frac{a^2}{c^2}\frac{c - b}{a - b}\,z;$$

therefore substituting in (1)

$$-\frac{a^2}{c^2}\frac{c-b}{a-b}x - \frac{b^2}{c^2}\frac{c-a}{b-a}x + x = 1,$$

$$\text{or } x\left(1 + \frac{a^2}{c^2}\frac{b-c}{a-b} + \frac{b^2}{c^2}\frac{c-a}{a-b}\right) = 1,$$

$$x\left\{c^2(a-b) + a^2(b-c) + b^2(c-a)\right\} = c^2(a-b),$$

$$x\left\{c^2(a-b) + ab(a-b) - c(a^2-b^2)\right\} = c^2(a-b).$$

Dividing by $a - b$

$$x\left\{c^2 + ab - c(a+b)\right\} = c^2,$$

$$\text{or } x(c-a)(c-b) = c^2;$$

$$\therefore x = \frac{c^2}{(c-a)(c-b)}.$$

In like manner $y = \dfrac{b^2}{(b-a)(b-c)}$, and $z = \dfrac{a^2}{(a-b)(a-c)}$.

The method employed in the following example is sometimes convenient, and is applicable to equations in which the sum of the indices of the unknown quantities is the same in each term.

Ex. 6.
$$x^2 + xy = 10 \qquad (1),$$
$$2x + y = 7 \qquad (2).$$

In a system of equations such as this, we may assume

$$y = mx;$$

$$\therefore x^2(1+m) = 10;$$

$$\text{and } x(2+m) = 7,$$

$$\text{or } x^2(2+m)^2 = 49;$$

$$\therefore 10(2+m)^2 = 49(1+m),$$

$$m^2 + 4m + 4 = \frac{49}{10}m + \frac{49}{10},$$

$$m^2 - \frac{9}{10}m + \frac{81}{400} = \frac{9}{10} + \frac{81}{400} = \frac{441}{400},$$

$$m = \frac{9 \pm 21}{20} = \frac{3}{2} \text{ or } -\frac{3}{5};$$

$$\therefore\ x = \frac{7}{2+m} = 2 \text{ or } 5,$$

$$y = 3 \text{ or } -3.$$

77. We have already given instances of the solution of a triple system of equations, Ex. 4 and 5: the general method of obtaining the value of any one of the unknown quantities at once from the equations, which we are now about to give, is worthy of notice.

Let the system of equations be as follows,

$$ax + by + cz = d \qquad (1),$$

$$a'x + b'y + c'z = d' \qquad (2),$$

$$a''x + b''y + c''z = d'' \qquad (3).$$

Multiply (1) by $b'c'' - b''c'$,

(2) by $b''c - bc''$,

(3) by $bc' - b'c$,

and add them all together; it will then be seen that the coefficients of $y$ and $z$ will be respectively zero, and we shall therefore have

$$x = \frac{d(b'c'' - b''c') + d'(b''c - bc'') + d''(bc' - b'c)}{a(b'c'' - b''c') + a'(b''c - bc'') + a''(bc' - b'c)}.$$

And the values of $y$ and $z$ may be written down in like manner. A little practice will make the student familiar with this method, which is called that of *Cross Multiplication* from the manner in which the multipliers of the three equations are formed.

We will illustrate the method by applying it to a set of equations already solved.

$$x + y + z = 0, \qquad (1)$$

$$x + 2y + 3z = 1, \qquad (2)$$

$$2x + y + 3z = 2. \qquad (3)$$

The multipliers will be seen to be $3, -2$, and $1$ respectively: hence

$$x = \frac{-2+2}{3-2+2} = 0,$$

the multipliers for $y$ are $-3$, $-1$, and $2$; and those for $z$, $-3$, $1$, and $1$; hence we have,

$$y = \frac{-1+4}{-3-2+2} = -1,$$

$$\text{and } z = \frac{1+2}{-3+3+3} = 1;$$

which are the results already obtained.

78. Sometimes a system of equations may be given which are not really sufficient to determine the unknown quantities, in consequence of not being *independent*, that is to say, in consequence of any one of the equations being deducible from the rest. This dependence is not always to be detected by inspection, but will become apparent if we endeavour to solve the equations.

Ex. Let it be required to determine $x$, $y$ and $z$ from the following system:

$$2x + 3y + z = 11 \qquad (1),$$

$$x - y + 2z = 5 \qquad (2),$$

$$x + 9y - 4z = 7 \qquad (3).$$

Multiplying (2) by 3, and adding the result to (1), we have

$$5x + 7z = 26 \qquad (4).$$

Again, multiplying (2) by 9 and adding (3),

$$10x + 14z = 52 \qquad (5).$$

The two equations which we have thus obtained, viz. (4) and (5), are identical, and therefore the given system is not sufficient to determine $x$, $y$ and $z$; in fact those three equations are equivalent to only *two* independent equations, any one of them being derivable from the other two.

The preceding examples must serve for the elucidation of the method of solving equations; the illustrations of the process might be indefinitely extended, but a familiar acquaintance with the most convenient methods can only be acquired by the practice of actual solution on the part of the student himself.

## ON PROBLEMS WHICH MAY BE RESOLVED BY MEANS OF ALGEBRAICAL EQUATIONS.

79. A vast variety of questions, which present great and perhaps insuperable difficulty to a mind unaided by the art of symbolical reasoning, are rendered extremely simple by reducing them to algebraical equations.

80. The most general rule which can be given for the solution of such questions is this: Denote the unknown quantities of the problem by symbols ($x$, $y$, $z$, &c.) and then express the conditions of the problem in terms of those symbols; we do by this means, in fact, express by algebraical sentences or equations the ordinary written sentences in which the problem is given. The equations thus constructed must be solved according to the methods which have been previously discussed; the equations may, for aught we can tell by inspection of the problem *à priori*, rise to a degree above the second, but of course we shall confine our attention here to those problems which produce either simple equations or quadratics.

Ex. 1. Divide the number 16 into two parts such that their difference shall be equal to half the number itself.

Let $x$ represent one of the parts, then will $16-x$ represent the other and $16-2x$ will represent the difference of the two; but, by the question, this difference is equal to $\frac{16}{2}$ or 8;

$$\begin{aligned} \therefore\ 16-2x &= 8, \\ 2x &= 8, \\ x &= 4 \quad \text{one of the parts,} \\ 16-x &= 12 \quad \text{the other part.} \end{aligned}$$

Ex. 2. Two pipes will separately fill a cistern in $a$ hours and $b$ hours respectively; in how long a time will they fill it together?

Let $x$ be the number of hours required.

Call the whole content of the cistern 1; then since the first

pipe pours in 1 (or the whole quantity required to fill the cistern) in $a$ hours, it pours in $\frac{1}{a}$ in one hour, and therefore $\frac{x}{a}$ in $x$ hours.

Similarly, the second pipe pours in $\frac{x}{b}$ in the same time; therefore they together pour in $\frac{x}{a} + \frac{x}{b}$.

But this quantity is the whole content of the cistern, or 1;

$$\therefore \frac{x}{a} + \frac{x}{b} = 1,$$

$$\text{and } x = \frac{1}{\frac{1}{a} + \frac{1}{b}} = \frac{ab}{a+b}.$$

Ex. 3. $A$'s money exceeds $B$'s and $C$'s by £$a$ and £$b$ respectively, and that of $B$ and $C$ together is £$c$: find the sum possessed by each.

Let $x$ = the sum possessed by $A$;

$\therefore x - a =$ ........................ $B$,

and $x - b =$ ........................ $C$;

therefore by question,

$$x - a + x - b = c,$$

$$2x = a + b + c,$$

$$x = \frac{a+b+c}{2} = A\text{'s},$$

$$x - a = \frac{-a+b+c}{2} = B\text{'s},$$

$$x - b = \frac{a-b+c}{2} = C\text{'s}.$$

Ex. 4. Divide £$a$ among three persons, so that the first may have $m$ times as much as the second, and the third $n$ times as much as the first and second together.

Let $x$ = the sum allotted to the second;

$\therefore mx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ first,

and $n(x + mx) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ third;

$$\therefore x + mx + n(x + mx) = a,$$

$$x(1 + m)(1 + n) = a,$$

$$x = \frac{a}{(1 + m)(1 + n)},$$

$$mx = \frac{ma}{(1 + m)(1 + n)},$$

$$n(x + mx) = \frac{na}{1 + n}.$$

Ex. 5. Divide the quantity $a$ into two parts, such that the product of the whole and one of the parts shall be equal to the square of the other part.

Let $x$ = one part;

$\therefore a - x$ = the other;

$\therefore$ by the question,

$$ax = (a - x)^2$$

$$= a^2 - 2ax + x^2,$$

$$x^2 - 3ax = -a^2;$$

completing the square,

$$x^2 - 3ax + \frac{9a^2}{4} = \frac{9a^2}{4} - a^2 = \frac{5a^2}{4};$$

$$\therefore x - \frac{3a}{2} = \pm \frac{\sqrt{5}}{2} a,$$

$$x = \frac{3 \pm \sqrt{5}}{2} a.$$

We must take the negative sign, because $\frac{3 + \sqrt{5}}{2} a$ is greater than $a$;

$$\therefore x = \frac{3 - \sqrt{5}}{2} a,$$

$$a - x = \frac{\sqrt{5} - 1}{2} a.$$

Ex. 6. The sum of two numbers is $a$, and the sum of their cubes $b$; find the numbers.

Let $x =$ one of the numbers;

$\therefore a - x =$ the other;

$\therefore$ by the question,

$$x^3 + (a - x)^3 = b,$$

$$\text{or } x^3 + a^3 - 3a^2x + 3ax^2 - x^3 = b,$$

$$x^2 - ax = \frac{b - a^3}{3a},$$

$$x^2 - ax + \frac{a^2}{4} = \frac{b - a^3}{3a} + \frac{a^2}{4};$$

$$\therefore x = \frac{a}{2} \pm \sqrt{\frac{b - a^3}{3a} + \frac{a^2}{4}},$$

$$a - x = \frac{a}{2} \mp \sqrt{\frac{b - a^3}{3a} + \frac{a^2}{4}}.$$

Hence the two numbers are $\frac{a}{2} + \sqrt{\frac{b - a^3}{3a} + \frac{a^2}{4}}$,

and $\frac{a}{2} - \sqrt{\frac{b - a^3}{3a} + \frac{a^2}{4}}$.

Ex. 7. There are three magnitudes, the sum of the first and second of which is $a$, that of the first and third $b$, and that of the second and third $c$; find them.

Let the magnitudes be represented by $x, y, z$ respectively; then,

$$x + y = a, \qquad x + z = b, \qquad y + z = c;$$

adding these equations and dividing the result by 2,

$$x + y + z = \frac{a + b + c}{2}.$$

$$\text{Hence } x = \frac{a + b + c}{2} - c = \frac{a + b - c}{2},$$

$$y = \frac{a + b + c}{2} - b = \frac{a - b + c}{2},$$

$$z = \frac{a + b + c}{2} - a = \frac{-a + b + c}{2}.$$

Ex. 8. Required four magnitudes the products of which taken three together are $a^3$, $b^3$, $c^3$, and $d^3$.

Call the magnitudes $x$, $y$, $z$, and $u$.

Then $yzu = a^3$, $xzu = b^3$, $xyu = c^3$, $xyz = d^3$.

Multiplying these equations together, we have

$$x^3y^3z^3u^3 = a^3b^3c^3d^3;$$

$$\therefore xyzu = abcd;$$

$$\therefore x = \frac{abcd}{yzu} = \frac{abcd}{a^3},$$

$$y = \frac{abcd}{b^3}, \quad z = \frac{abcd}{c^3}, \quad u = \frac{abcd}{d^3}.$$

## ON RATIOS.

81. Ratio is the relation which quantities of the same kind bear to each other in respect of magnitude.

Thus 6 is twice as great as 3, and 2 is twice as great as 1; therefore we should say that the ratio of 6 to 3 is the same as that of 2 to 1; or we may write for shortness' sake,

$$6 : 3 :: 2 : 1.$$

In speaking of the ratio of two quantities $a : b$, $a$ and $b$ are called the *terms* of the ratio, and $a$ is distinguished as the *antecedent*, $b$ as the *consequent*.

82. It is easy to shew that the terms of a ratio may be multiplied or divided by any (the same) number, without affecting the value of the ratio. For distinctness' sake let us consider $a$ and $b$ as representing two *lines*, then by the symbol $a$ we mean to denote a line $a$ times as great as a certain standard line (a *foot* for instance), and by $b$ a line $b$ times as great. The lines in question are then in the proportion of the numbers $a$, $b$; but if we had taken a line only half as long (six inches) for the standard, the lines would have been represented by $2a$, and $2b$; but their ratio of course is not altered;

$$\therefore a : b :: 2a : 2b.$$

In like manner it would appear that $a : b :: 3a : 3b$, and generally that the terms of a ratio may be multiplied by any number without affecting the value of the ratio.

Conversely, the terms may be *divided* by any number.

83. *Hence it follows that we may represent ratios algebraically by fractions, of which the antecedent is the numerator and the consequent the denominator.*

For by what has been said the ratio of $a : b$ is the same as the ratio of $\frac{a}{b} : 1$; now 1 is a given quantity, and therefore we may take the fraction $\frac{a}{b}$ as the symbol of the ratio $\frac{a}{b} : 1$, and therefore as the representative of the ratio $a : b$.

In fact, the ratio $a : b$ may be conceived to mean that $a$ is as many times as great as $b$, as $\frac{a}{b}$ is as great as 1, and therefore the magnitude of the fraction $\frac{a}{b}$ *measures* the magnitude of the ratio of $a : b$.

Henceforth therefore we shall represent the ratio $a : b$ by the fraction $\frac{a}{b}$.

If $a$ is greater than $b$, the ratio $\frac{a}{b}$ is called a ratio of *greater inequality*.

If $a = b$, the ratio is called a ratio of *equality*.

If $a$ is less than $b$, a ratio of *less inequality*.

84. *A ratio of greater inequality is diminished, and of less inequality increased, by adding the same quantity to each of its terms.*

Let $\frac{a}{b}$ be any ratio, and let $x$ be added to each of its terms; then

$$\frac{a+x}{b+x} \text{ is } > \text{ or } < \frac{a}{b},$$

according as

$$(a+x)\,b \text{ is } > \text{ or } < a\,(b+x),$$
$$\text{or as } bx \text{ is } > \text{ or } < ax,$$
$$\text{or as } b \text{ is } > \text{ or } < a,$$

i.e. as the ratio is one of less or greater inequality.

85. Ratios are compounded by multiplying together their corresponding terms.

Thus $\frac{a}{b}$ compounded with $\frac{c}{d}$ becomes $\frac{ac}{bd}$.

According to the method of treating ratios adopted in Euclid's elements, when there are any number of magnitudes of the same kind, the first is said to have to the last of them, the ratio compounded of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on unto the last magnitude. For example, if $a$, $b$, $c$, $d$ be four magnitudes of the same kind, the ratio of $a : d$ is said to be compounded of the three ratios $a : b$, $b : c$, $c : d$. It will be easily seen that this definition coincides with that which has just been given of the method of compounding ratios; for representing the ratio of $a : b$ by $\frac{a}{b}$, and that of $b : c$ by $\frac{b}{c}$, the ratio compounded of the two ratios $a : b$ and $b : c$ will be according to our definition represented by $\frac{a}{b} \times \frac{b}{c}$ or by $\frac{a}{c}$, that is, by $a : c$ which is the compound ratio according to Euclid's definition.

86. *A ratio is increased by being compounded with another of greater inequality, and diminished by being compounded with one of less.*

For $\frac{a}{b}$ compounded with $\frac{c}{d}$ becomes $\frac{ac}{bd}$;

$$\text{but } \frac{ac}{bd} \text{ is } > \text{ or } < \frac{a}{b},$$

$$\text{according as } \frac{c}{d} \text{ is} > \text{or} < 1,$$

$$\text{or as } c \text{ is} > \text{or} < d,$$

which proves the proposition.

## ON PROPORTION.

87. Proportion is the equality of ratios; and therefore, algebraically, four quantities are said to be proportional, when the fraction expressing the ratio of the first and second is equal to that expressing the ratio of the third and fourth; that is, $a : b :: c : d$ when $\frac{a}{b} = \frac{c}{d}$.

88. *If* $a : b :: c : d$, *then* $ad = bc$.

For, as we have just seen,

$$\frac{a}{b} = \frac{c}{d},$$

$$\text{and } \therefore ad = bc.$$

89. *If* $a : b :: c : d$, *then* $a \pm b : b :: c \pm d : d$.

$$\text{For, } \frac{a}{b} = \frac{c}{d}; \ \therefore \frac{a}{b} \pm 1 = \frac{c}{d} \pm 1,$$

$$\text{or } \frac{a \pm b}{b} = \frac{c \pm d}{d}.$$

90. *If* $a : b :: c : d$, *then*

$ma \pm nb : pa \pm qb :: mc \pm nd : pc \pm qd.$

$$\text{For, } \frac{a}{b} = \frac{c}{d};$$

$$\therefore \frac{ma}{nb} = \frac{mc}{nd},$$

$$\frac{ma}{nb} \pm 1 = \frac{mc}{nd} \pm 1,$$

$$\frac{ma \pm nb}{nb} = \frac{mc \pm nd}{nd},$$

$$\frac{ma \pm nb}{mc \pm nd} = \frac{nb}{nd} = \frac{b}{d};$$

in like manner it may be shewn that

$$\frac{pa \pm qb}{pc \pm qd} = \frac{b}{d};$$

$$\therefore \frac{ma \pm nb}{mc \pm nd} = \frac{pa \pm qb}{pc \pm qd}.$$

91. *If* a : b :: c : d :: e : f :: &c., *then*
a : b :: a + c + e + ... : b + d + f + ...

$$\text{For } \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots;$$

$$\therefore ad = bc,$$
$$af = be,$$
$$\&c. = \&c.$$

∴ by addition,

$$ad + af + \ldots = bc + be + \ldots,$$

$$\therefore ab + ad + af + \ldots = ba + bc + be + \ldots,$$

$$\text{or } a\{b + d + f + \ldots\} = b\{a + c + e + \ldots\};$$

$$\therefore \frac{a}{b} = \frac{a + c + e + \ldots\ldots}{b + d + f + \ldots\ldots}.$$

92. A variety of other propositions in proportion may be demonstrated in like manner as the preceding. The greatest simplicity is introduced by this method of representing ratios by fractions, and it will be instructive to inquire into the reason of the much more complicated processes, which Euclid has found it necessary to employ in the fifth book of his Elements.

Euclid's definition of proportion is this: Four quantities are said to be proportional, when any equimultiples whatever being taken of the first and third, and any whatever of the second and fourth; if the multiple of the first is greater than

that of the second, the multiple of the third is greater than that of the fourth, if equal equal, and if less less. Now this definition is an immediate consequence of the algebraical representation of ratio; for suppose $a : b :: c : d$,

$$\text{then } \frac{a}{b} = \frac{c}{d},$$

$$\text{and } \frac{pa}{qb} = \frac{pc}{qd},$$

$$\text{and if } pa \text{ is } > qb,\ pc \text{ is } > qd,$$

$$\text{if } pa = qb,\ pc = qd,$$

$$\text{and if } pa < qb,\ pc < qd,$$

which is in accordance with Euclid's definition.

And, conversely, from Euclid's definition may be deduced the algebraical rule of proportion; that is, we can shew that if, by that definition, $a : b :: c : d$, then must $\frac{a}{b} = \frac{c}{d}$.

For let $a, b, c, d$ be four quantities such that if any equimultiples $pa$, $pc$ be taken of $a$ and $c$, and any equimultiples $qb$, $qd$ of $b$ and $d$, if $pa$ be $> qb$, $pc$ is $> qd$, if equal equal, and if less less.

Then since we may choose $p$ and $q$ as we please, we can make $\frac{pa}{qb}$ as nearly equal to 1 as we please; we cannot, it is true, always make it precisely equal to 1, because $p$ and $q$ are to be whole numbers, and the ratio of $a$ to $b$ is not necessarily expressible by the ratio of two whole numbers; but nevertheless we can make the fraction $\frac{pa}{qb}$ *as near to unity as we please**; we may therefore suppose that we have taken $p$ and $q$ such that $\frac{pa}{qb} = 1$, since this equation can be satisfied to any assignable degree of accuracy; in other words, $\frac{pa}{qb}$ can be made to differ from 1 by a quantity less than any assignable quantity.

* Suppose, for example, we have such a quantity as $\sqrt{2}$, then we can make $\frac{p}{q}\sqrt{2}$ as near to unity as we please; for $\sqrt{2} = 1.41421...$, and therefore, $\frac{100000}{141421}\sqrt{2} = 1$ nearly, and we can make the approximation as much nearer as we please by taking a greater number of figures in the square root.

but by definition we must in this case have also

$$\frac{pc}{qd} = 1;$$

$$\therefore \frac{pa}{qb} = \frac{pc}{qd},$$

$$\text{or } \frac{a}{b} = \frac{c}{d}.$$

Hence it appears that Euclid's definition of proportion and the algebraical follow each from the other; but wherein is the propriety of Euclid's peculiar definition? In this, that the algebraical test is not applicable to geometrical quantities; we can represent addition and subtraction geometrically, but not *division*, and therefore it is necessary in geometrical investigations to adopt some definition which involves only the notion of addition and of a comparison of magnitudes with reference to *greater* or *less*.

## ON VARIATION.

93. When one quantity $y$ depends upon another $x$, in such a manner that, if $x$ is changed in value, the value of $y$ is changed in the same proportion, then $y$ is said to *vary directly* as $x$, or shortly, to *vary* as $x$.

For instance, we know by Euclid, VI. 1, that if we double the base of a triangle the vertex remaining the same, we double the area, and that in whatever proportion we alter the base the area is altered in the same proportion, hence we should say that (the altitude being given,) the area of a triangle *varies as the base*.

The phrase *$y$ varies as $x$* is written thus, $y \propto x$.

The student will observe that the word *vary* is here used in a peculiar technical sense, and that it does not imply mere change of value.

94. It will be seen that we have here introduced the notion of quantities entirely different from those hitherto considered; hitherto we have had to do only with quantities which have some determinate value, but the relation between $y$ and $x$ implied in the fact of $y$ varying as $x$ does not

determine either $x$ or $y$, but only a relation between them. Quantities of this kind we call *variable* quantities, to distinguish them from others the value of which is determinate and which we call *constant*.

95. The relation expressed by $y \propto x$ is equivalent to the equation $y = Cx$, where $C$ is some constant quantity; for $\frac{y}{x}$ is the ratio of $y$ to $x$, and the preceding equation expresses that this is constant, or always the same whatever values $x$ and $y$ may have; and this is the same thing as saying, that when one is increased the other is increased in the same proportion.

96. If we have any two corresponding values of $x$ and $y$ given we can determine the quantity $C$; thus, suppose $y \propto x$ and it is given that when $x = 1$, $y = 2$, then we have

$$y = Cx,$$

$$\text{but } 2 = C1,$$

$$\therefore\ y = 2x.$$

97. When two quantities are connected by the relation $y = \frac{C}{x}$, $y$ is said to vary *inversely* as $x$.

And when three quantities $x, y, z$ are connected in such a manner that $z = Cxy$, $z$ is said to vary jointly as $x$ and $y$.

98. *If* $y \propto x$, and $z \propto y$, *then*, $z \propto x$.

For let
$$y = Cx,$$
$$z = C'y;$$
$$\therefore\ z = CC'.x,$$
and $CC'$ is constant; $\therefore\ z \propto x$.

99. *If* $y \propto x$, *and* $z$ *also* $\propto x$, then $\sqrt{yz} \propto x$.

For let
$$y = Cx,$$
$$z = C'x;$$
$$\therefore\ yz = CC'.x^2,$$
$$\sqrt{yz} = \sqrt{CC'}.x,$$
$$\text{and } \sqrt{CC'} \text{ is constant, } \therefore\ \sqrt{yz} \propto x.$$

Many other propositions may be demonstrated in like manner with perfect facility. We shall conclude with the following proposition.

100. *If* z *be a quantity depending upon two others,* x *and* y, *in such a manner that when* x *is constant and* y *allowed to vary* $z \propto y$, *and when* y *is constant and* x *allowed to vary* $z \propto x$, *then when* x *and* y *both vary* z *will* $\propto xy$.

Let $z = u.xy$, where $u$ is a quantity which, for anything we know at present to the contrary, may involve $x$ or $y$ or both.

Then when $x$ is constant and $y$ variable $z \propto y$; but $z = ux.y$, therefore $ux$ does not involve $y$, or $u$ does not involve $y$.

In like manner $u$ does not involve $x$, therefore it is constant, or $z \propto xy$.

We may illustrate the preceding proposition as follows: When the base of a triangle is given the area $\propto$ the altitude, and when the altitude is given the area $\propto$ the base; hence when neither is given, the area $\propto$ base $\times$ altitude.

The same method of demonstration is applicable to the following more general proposition:

*If* z *be a quantity depending upon* n *others, in such a manner that when any* $n - 1$ *of them are constant* z *varies as the remaining one; then when all the* n *quantities vary,* z *varies as their product.*

## ON ARITHMETICAL PROGRESSION.

101. Def. Quantities are said to be in arithmetical progression, when they increase or decrease by a common difference.

Thus $a, a + d, a + 2d, \ldots\ldots\ldots$ is an arithmetical series.

102. *To sum an arithmetical series.*

Let $a$ be the first term, $d$ the common difference of the terms; then the second term will be $a + d$, the third $a + 2d$, and generally the $n^{\text{th}}$ term will be $a + (n - 1) d$.

Let $S$ be the sum of $n$ terms, then

$$S = a + a + d + a + 2d + \ldots\ldots\ldots + a + (n - 1) d;$$

writing the terms in the reverse order, we have

$$S = a + (n-1)\,d + a + (n-2)\,d + a + (n-3)\,d + \ldots\ldots + a;$$

adding together these two equations,

$$2S = 2a + (n-1)\,d + 2a + (n-1)\,d + 2a + (n-1)\,d \ldots\ldots\ldots$$
$$+\, 2a + (n-1)\,d,$$

$$= \{2a + (n-1)\,d\}\,n, \text{ since there are } n \text{ terms};$$

$$\therefore\ S = \{2a + (n-1)\,d\}\,\frac{n}{2},$$

which is the expression for the sum required.

The expression for $S$ may also be written thus: let $l$ be the *last* term, i.e. $l = a + (n-1)\,d$, then

$$S = (a + l)\,\frac{n}{2}.$$

COR. Any three of the quantities $a$, $d$, $n$, and $S$, being given, the fourth may be found.

Ex. 1. Find the sum of 10 terms of the arithmetical series 2, 5, 8, .........

Here $\qquad a = 2,\ \ d = 3,\ \ n = 10;$

$$\therefore\ S = (4 + 9 \times 3)\,5$$
$$= 31 \times 5 = 155.$$

Ex. 2. There is an arithmetical series the fourth term of which is 9 and the seventh 15: find the series.

The formula for the $n^{\text{th}}$ term is $a + (n-1)\,d$, therefore in this example we have

$$a + 3d = 9,$$
$$a + 6d = 15.$$

Subtracting the first of these equations from the second,

$$3d = 6,$$
$$d = 2;$$
$$\therefore\ a = 9 - 6 = 3;$$

and the series is 3, 5, 7, 9.........

Ex. 3. Insert $n$ arithmetical means between $a$ and $b$. This is in other words to form an arithmetical series of $n + 2$ terms, of which the first shall be $a$ and the last $b$.

Let $d$ be the common difference, then we must have

$$a + (n + 1) d = b;$$

$$\therefore d = \frac{b - a}{n + 1}.$$

Hence the terms required will be

$$a + \frac{b - a}{n + 1}, \ a + 2\frac{b - a}{n + 1}, \ \&c.$$

Ex. 4. Find an arithmetical series in which the seventh term is three times as great as the second, and the fourth exceeds the second by four.

If $a$ be the first term, $d$ the common difference, we have the conditions

$$a + 6d = 3(a + d),$$
$$a + 3d = a + d + 4,$$
$$\text{or } \ 2a = 3d,$$
$$2d = 4;$$
$$\therefore d = 2, \ a = 3,$$

and the series is 3, 5, 7, 9, &c.

## ON GEOMETRICAL PROGRESSION.

103. Def. A series of quantities are said to be in geometrical progression when each term of the series is equal to that which precedes it multiplied by some constant factor, *i. e.* some factor which is the same for all the terms, or in other words, when the ratio of any two successive terms is the same.

Thus $a, ar, ar^2, ar^3, \ldots\ldots\ldots$ is a geometrical series.

104. *To sum a geometrical series.*

Let $a$ be the first term, $r$ the common ratio of the terms; then the second term will be $ar$, the third $ar^2$, and generally the $n^{th}$ term will be $ar^{n-1}$.

Let $S$ be the sum of $n$ terms, then

$$S = a + ar + ar^2 + \ldots\ldots\ldots + ar^{n-1},$$

multiplying by $r$ we have

$$rS = ar + ar^2 + \dots\dots\dots + ar^{n-1} + ar^n;$$

subtracting the former of these equations from the latter, we have

$$(r-1)S = ar^n - a;$$

$$\therefore\ S = a\frac{r^n - 1}{r - 1},$$

which is the expression for the sum required. The result may be easily verified by actual division.

COR. Any three of the four quantities $a$, $r$, $n$, $S$ being given, the fourth may be found.

105. The formula of the preceding article may be written thus;

$$S = \frac{a}{1-r} - \frac{ar^n}{1-r};$$

the first term of this expression remains the same whatever value be given to $n$, the second increases or decreases as $n$ increases according as $r$ is greater or less than 1. Suppose $r$ to be less than 1; then if $n$ be very great, $\frac{ar^n}{1-r}$ will be very small, and $S$ will not differ much from $\frac{a}{1-r}$; by making $n$ still greater $S$ will become still more nearly equal to $\frac{a}{1-r}$, but however large $n$ may be, $\frac{a}{1-r}$ is a quantity which $S$ will never precisely reach, though it may be made to approximate to it by less than any assignable quantity. Hence $\frac{a}{1-r}$ may be spoken of as the *limit* to which the series approaches indefinitely when the number of its terms is indefinitely increased; the *limit* of the series is more commonly expressed by the phrase the "sum of the series continued *ad infinitum*," a phrase which though not strictly accurate may be conveniently used if the sense attached to it be carefully borne in mind.

When a limit can thus be found, to which the sum of a series continually approaches as the number of its terms is increased, and from which its value can be made to differ by less than any assignable quantity, the series is said to be a *convergent* series. Thus we should say that in a geometrical series the condition of convergency is that $r$ shall be less than 1; if $r$ be greater than 1, the series will be *divergent*.

106. An example of a geometrical series continued *ad infinitum* occurs in arithmetic, in the case of recurring decimals: thus the recurring decimal $.333\ldots\ldots = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \ldots\ldots$,

and the sum of this series is $\dfrac{\frac{3}{10}}{1 - \frac{1}{10}} = \frac{3}{9} = \frac{1}{3}$.

But more generally

107. *To find the vulgar fraction corresponding to a given recurring decimal.*

Let the decimal be represented by $A.BRRR$*......... where $A$ is the integral part, $B$ the nonrecurring decimal part, and $R$ the recurring; and suppose $B$ to contain $p$ digits, and $R$ to contain $q$ digits.

$$\text{Let } S = A.BRRR\ldots\ldots\ldots;$$
$$\therefore\ 10^{p+q} S = ABR.RR\ldots\ldots\ldots,$$
$$\text{and } 10^p S = AB.RR\ldots\ldots\ldots;$$
$$\therefore\ (10^{p+q} - 10^p)\, S = ABR - AB,$$
$$\text{and } S = \frac{ABR - AB}{10^p(10^q - 1)}.$$

This quantity reduced to its lowest terms will be the vulgar fraction required.

Ex. 1. Find the sum of 10 terms of the geometrical series 1, 2, 4, 8, .........

* It is hardly necessary to observe, that $BR$ does not here stand for $B \times R$ according to algebraical usage; for instance, if the decimal were 19.31263263...... we should have $A = 19$, $B = 31$, $R = 263$.

In this case $a = 1$, $r = 2$, $n = 10$;

$$\therefore S = \frac{2^{10} - 1}{2 - 1} = 2^{10} - 1 = 1023.$$

Ex. 2. There is a geometrical series of which the *second* term is 6 and the *fourth* 54; find it.

Here
$$ar = 6,$$
$$ar^3 = 54;$$
$$\therefore r^2 = \frac{54}{6} = 9,$$
$$r = \pm 3;$$
$$\therefore a = \frac{6}{\pm 3} = \pm 2;$$

$\therefore$ the series is

$$2, 6, 18, 54, \ldots\ldots\ldots$$
$$\text{or } -2, 6, -18, 54, \ldots\ldots\ldots$$

Ex. 3. Insert $n$ geometrical means between $a$ and $b$. This is in other words to construct a series of which the first term is $a$, and the $(n+2)^{\text{th}}$ term $b$.

Therefore we must have

$$ar^{n+1} = b,$$
$$\therefore r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}},$$

and the geometrical means required are

$$a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, \quad a\left(\frac{b}{a}\right)^{\frac{2}{n+1}} \ldots\ldots\ldots\ldots\ a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}.$$

Ex. 4. Find the sum of the series $1 + \frac{1}{2} + \frac{1}{4} + \ldots$ *ad infinitum*.

In this case
$$a = 1,\ r = \tfrac{1}{2};$$
$$\therefore S = \frac{1}{1 - \frac{1}{2}} = 2.$$

Ex. 5. Find the vulgar fraction corresponding to the recurring decimal $2.46\overline{262}\ldots\ldots$

Let
$$S = 2.46\overline{62},$$
$$1000\,S = 2462.\overline{62},$$
$$10\,S = 24.\overline{62};$$
$$\therefore 990\,S = 2438,$$
$$\text{and } S = \frac{2438}{990} = \frac{1219}{495}.$$

## ON HARMONICAL PROGRESSION.

108. DEF. A series of quantities are said to be in harmonical progression, when any three successive terms are so related, that the first is to the third as the difference between the first and the second is to the difference between the second and third.

Thus if $a$, $b$, $c$ are in harmonical progression,

$$a : c :: a - b : b - c.$$

*The reciprocals* of quantities in harmonical progression are in arithmetical progression.*

Let $a$, $b$, $c$ be quantities in harmonical progression, then by definition,

$$\frac{a}{c} = \frac{a-b}{b-c},$$

$$\text{or } \frac{b-c}{c} = \frac{a-b}{a},$$

$$\text{or } \frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a},$$

which proves that the difference between $\frac{1}{a}$ and $\frac{1}{b}$ is the same as between $\frac{1}{b}$ and $\frac{1}{c}$, or that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in arithmetical progression.

COR. Hence we may, if we please, take it as the definition of quantities in harmonical progression that their reciprocals are in arithmetical.

* The *reciprocal* of a quantity is unity divided by that quantity: thus $\frac{1}{a}$ is the *reciprocal* of $a$.

109. A series of quantities in harmonical progression admits of no simple summation.

110. The three kinds of progression which have been treated of, may be brought under one point of view as follows:

If $a$, $b$, $c$ are in *arithmetical* progression, we have

$$\frac{a-b}{b-c}=\frac{a}{a}.$$

If in *geometrical*,

$$\frac{a-b}{b-c}=\frac{a}{b}.$$

If in *harmonical*,

$$\frac{a-b}{b-c}=\frac{a}{c}.$$

## ON PERMUTATIONS AND COMBINATIONS.

111. The different ways in which any number of quantities can be arranged are called their *permutations*.

Thus the *permutations* of the letters, $a$, $b$, $c$ taken two together are $ab$, $ac$, $ba$, $bc$, $ca$, $cb$.

The *combinations* of a number of quantities are the collections which can be made of them without regard to arrangement.

Thus the *combinations* of $a$, $b$, $c$ taken two together, are $ab$, $ac$, $bc$: $ab$, $ba$, which were *two permutations*, form only *one combination*, and so of the rest.

112. *To find the number of permutations of* n *things taken* r *together.*

Let $\{nPr\}$ denote the number of permutations of $n$ things $a$, $b$, $c$, $d$,...taken $r$ together.

Then it is manifest that the number of permutations of $n$ things taken 1 together is $n$, or $\{nP1\}=n$.

Again, to find the number of permutations of $n$ things taken two together, we observe that $a$ may be placed before each of the $n-1$ other letters $b$, $c$, $d$..., thus forming $n-1$ permutations in which $a$ stands first; the same may be said

and so of the other letters; therefore the whole number of permutations which can be formed is $n(n-1)$,

$$\text{or } (nP2) = n(n-1).$$

Now suppose that $(nPr) = n(n-1)(n-2)\ldots(n-r+1)$; then if we omit $a$ and form the permutations of $n-1$ things taken $r$ together we shall have $(n-1)(n-2)\ldots(n-r)$ of such permutations (by writing $n-1$ for $n$ in the preceding formula), and before each of these we can place $a$, thus forming $(n-1)(n-2)\ldots(n-r)$ permutations of $n$ things taken $r+1$ together in which $a$ stands first; the same may be said of each of the other letters, and therefore

$$(nPr+1) = n(n-1)(n-2)\ldots(n-r).$$

Hence if the formula assumed for $(nPr)$ be true for one value of $r$, it is true for the next superior value; but it is true when $r=2$, as we have seen, therefore it is true when $r=3$, therefore when $r=4$, therefore &c. therefore generally true.

The preceding proof is an instance of that process of induction which we have already used in several instances.

(Note.) The proof may be exhibited rather more concisely as follows.

Let $(nPr)$ denote the number of permutations of $n$ things $a$, $b$, $c$, $d$, ... taken $r$ together, as before. Now suppose we omit one of the letters, as $a$, and form the remainder into permutations taken $r-1$ together, of which, according to our notation, there will be $\{(n-1)P(r-1)\}$; then before each of these permutations we may place $a$, thus forming $\{(n-1)P(r-1)\}$ *permutations of $n$ things taken $r$ together, in which $a$ stands first*: the same may be said of $b$, $c$, $d$, and there are $n$ of them; therefore we shall have

$$\{nPr\} = n\,\{(n-1)P(r-1)\};$$

in like manner,

$$\{(n-1)P(r-1)\} = (n-1)\,\{(n-2)P(r-2)\},$$
$$\{(n-2)P(r-2)\} = (n-2)\,\{(n-3)P(r-3)\},$$
$$\&c. = \&c.$$
$$\{(n-r+2)P2\} = (n-r+2)\,\{(n-r+1)P1\}$$
$$= (n-r+2)(n-r+1),$$

[since it is manifest that $\{(n-r+1)P1\} = n-r+1$].

Now multiplying together the corresponding sides of these equations and leaving out the common factors, we have

$$\{nPr\} = n(n-1)(n-2) \ldots\ldots (n-r+2)(n-r+1).$$

Cor. If $r = n$, we have

$$\{nPn\} = n(n-1)(n-2) \ldots\ldots 2.1.$$

Ex. 1. Find the number of permutations of 7 things taken 4 together.

In this case $n = 7$, $r = 4$.

$$\therefore \ \{7P4\} = 7.6.5.4 = 840.$$

Ex. 2. Find the number of permutations of 5 things taken all together.

$$n = 5, \ r = 5.$$

$$\therefore \ \{5P5\} = 5.4.3.2.1 = 120.$$

Ex. 3. Determine the number of triliteral words, which can be formed of 8 consonants and 1 vowel, the vowel being always the central letter.

It is evident that the number required $= \{8P2\} = 8.7 = 56$.

113. *To find the number of permutations of* n *things taken all together, when a are of the same kind.*

Let $x$ be the number required.

Then since *all* the $a$ quantities enter into *each* permutation, if we suppose them all different, *each* permutation would be resolved into $\{aPa\}$ permutations, and therefore the whole number of permutations would be $\{aPa\}$ times as great; but in this case the number of permutations would be that of $n$ things, all different, taken all together, or $\{nPn\}$; hence we have

$$x\{aPa\} = \{nPn\},$$

$$\text{or} \quad x = \frac{n(n-1) \ldots\ldots 2.1}{a(a-1) \ldots\ldots 2.1}.$$

Cor. In like manner, if there were $a$ quantities of one kind, $\beta$ of another, $\gamma$ of another, &c. we should have for the number of permutations

$$\frac{n(n-1) \ldots\ldots 2.1}{1.2 \ldots a.1.2 \ldots \beta.1.2 \ldots \gamma.\&c.}$$

Ex. Determine the number of different arrangements of the letters forming the word *effect*.

In this example $n = 6$, $\alpha = 2$, $\beta = 2$;

$$\therefore \text{ the number required} = \frac{6 . 5 . 4 . 3 . 2 . 1}{1 . 2 . 1 . 2} = 180.$$

114. *To find the number of combinations of* n *things taken* r *together.*

Let $\{nCr\}$ denote the number of combinations. Then since the order of the quantities is not regarded in a combination, each combination of $r$ quantities may be resolved by permuting them into $\{rPr\}$ permutations. Hence we have

$$\{nCr\} \times \{rPr\} = \{nPr\},$$

$$\text{or } \{nCr\} = \frac{n(n-1) \ldots\ldots (n-r+1)}{1 . 2 \ldots\ldots\ldots\ldots r}.$$

Ex. 1. Find the number of combinations of 9 things taken 5 together.

In this case $n = 9$, $r = 5$;

$$\therefore \quad \{nCr\} = \frac{9 . 8 . 7 . 6 . 5}{1 . 2 . 3 . 4 . 5} = 126.$$

Ex. 2. How many different sums of money can be formed by selecting 3 coins from a heap containing a sovereign, a half-sovereign, a crown, a half-crown, a shilling, and a sixpence?

$$n = 6, \ r = 3;$$

$$\therefore \text{ the number required} = \{6C3\} = \frac{6 . 5 . 4}{1 . 2 . 3} = 20.$$

115. *The number of combinations of* n *things taken* r *together is the same as that of* n *things taken* (n − r) *together.*

In other words, $\{nCr\} = \{nC(n-r)\}$.

$$\text{Now } \{nCr\} = \frac{n(n-1)\ldots\ldots(n-r+1)}{1 . 2 \ldots\ldots\ldots\ldots r},$$

$$\{nC(n-r)\} = \frac{n(n-1)\ldots\ldots(r+1)}{1 . 2 \ldots\ldots\ldots (n-r)}.$$

$$\therefore \frac{\{nCr\}}{\{nC(n-r)\}} = \frac{n(n-1)\ldots\ldots(n-r+1)}{1 . 2 \ldots\ldots r} \cdot \frac{1 . 2 \ldots\ldots (n-r)}{n . (n-1) \ldots\ldots (r+1)}$$

$$= \frac{n(n-1)\ldots\ldots(n-r+1)(n-r)\ldots\ldots 2.1}{1.2\ldots\ldots r(r+1)\ldots\ldots(n-1)n}$$

$$= 1;$$

or $\{nCr\} = \{nC(n-r)\}$.

115 (bis). Another mode of proving this proposition is as follows. Whenever $r$ of the $n$ quantities are taken to form a combination, $n - r$ are necessarily omitted, and these may be supposed to be formed into a combination which may be called with reference to the former a *complementary* combination. Hence each combination of $r$ quantities has its complementary combination of $n - r$, and therefore the number of the two sets of combinations is equal.

Suppose for instance that we have the five letters $a, b, c, d, e$; if we form the combination *abc*, there will be a complementary combination *de*; for *abd*, there will be *ce*, and so on. Therefore for each combination of 3 letters there is a complementary combination of 2 letters; in other words $(5C3) = (5C2)$.

116. *To find the value of* r *for which the number of combinations of* n *things taken* r *together will be greatest.*

We have seen that $\{nCr\} = \dfrac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{1.2.3\ldots\ldots\ldots r}$,

and that $\{nC(r-1)\} = \dfrac{n(n-1)n-2\ldots\ldots(n-r+2)}{1.2.3\ldots\ldots\ldots(r-1)}$, so that $\{nCr\} = \{nC(r-1)\}\dfrac{n-r+1}{r}$. Hence if we suppose $r$ to become 1.2.3...successively, the number of combinations will continue to increase as long as $\dfrac{n-r+1}{r}$ is greater than 1, and therefore if we determine the value of $r$ for which $\dfrac{n-r+1}{r}$ first becomes equal to or less than 1 we shall be able to ascertain the value of $r$ for which $\{nCr\}$ is greatest.

Suppose $\dfrac{n-r+1}{r} = 1$, $\therefore r = \dfrac{n+1}{2}$; if $n$ be odd, this will be integral and will be the required value of $r$; if $n$ be even, the

value $\frac{n}{2}$ will be the last for which $\frac{n-r+1}{r}$ is greater than 1, and therefore will be the required value.

It may be observed that if $n$ be odd $\left\{nC\frac{n+1}{2}\right\} = \left\{nC\frac{n-1}{2}\right\}$ by Art. 115 so that there are two equal maximum values of $\{nCr\}$.

116 (bis). The preceding proposition may perhaps be made clearer by the following considerations.

If we take $n$ things and form them into combinations, first singly, then two together, then three, and so on, we shall obtain the following series of numbers,

$$n, \frac{n(n-1)}{1.2}, \frac{n(n-1)(n-2)}{1.2.3}, \ldots\ldots \frac{n(n-1)(n-2)}{1.2.3}, \frac{n(n-1)}{1.2}, n, 1.$$

And at the commencement of this series the terms will increase; but just as they increase at the beginning of the series, so will they decrease at its conclusion, since by the proposition of Art. 115 the terms will be the same at the beginning and end, with the exception of the last term of all or $\{nCn\}$, which has for its value 1

Hence the terms must attain a maximum value and then decrease; and if the number of terms be odd (i.e. if $n$ be odd) there will evidently be two equal maximum terms, if even there will be only one.

Thus the existence of a maximum number of combinations is seen to be an immediate result of the proposition that $\{nCr\} = \{nC(n-r)\}$.

Ex. To find the greatest number of combinations which can be formed of 9 things.

In this case $n = 9$, and $r = \frac{n+1}{2} = 5$, or $r = \frac{n-1}{2} = 4$.

$\therefore \frac{9.8.7.6}{1.2.3.4} = 126$ is the greatest number of combinations.

Let us verify this result

$$\{9C1\} = 9,$$

$$\{9C2\} = \frac{9.8}{1.2} = 36,$$

$$\{9C3\} = \frac{9.8.7}{1.2.3} = 84,$$

$$\{9\,C\,4\} = \frac{9.8.7.6}{1.2.3.4} = 126,$$

$$\{9\,C\,5\} = \frac{9.8.7.6.5}{1.2.3.4.5} = 126,$$

$$\{9\,C\,6\} = \frac{9.8.7.6.5.4}{1.2.3.4.5.6} = 84,$$

which shews that $\{9\,C\,4\}$ or $\{9\,C\,5\}$ is the greatest number of combinations, as determined by the rule.

## ON THE BINOMIAL THEOREM.

117. We have already seen (Art. 46) that $(a+b)^2 = a^2 + 2ab + b^2$, and that $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$; and we might find the expansion of any other positive integral power of $a+b$ by actual multiplication: the binomial theorem is a formula for the general expansion of $(a+b)^n$ according to powers of $b$, and that not only in the case of $n$ being positive and integral, but also when it is fractional and negative.

118. *To investigate the Binomial Theorem in the case of a positive integral index.*

We have by actual multiplication

$$(x+a)(x+b) = x^2 + (a+b)\,x + ab,$$

$$(x+a)(x+b)(x+c) = x^3 + (a+b+c)\,x^2 + (ab+ac+bc)\,x + abc.$$

In examining the preceding expressions we observe the following laws:

(1) That they consist of a series of descending powers of $x$, and that the first and highest index is the number of factors forming the expression.

(2) That the coefficient of the first term is unity; of the second, the sum of the products of the quantities $a, b, c$, taken *one* together; of the third, the sum of the products of the same taken *two* together; of the last, the product of them taken all together.

Let us suppose that this law holds for $n$ factors, that is, that

$$(x+a)(x+b)(x+c)\ldots(x+p) = x^n + S_1 x^{n-1} + S_2 x^{n-2} + \ldots + S_n$$

$$\begin{aligned} \text{where } S_1 &= a + b + c + \ldots + p, \\ S_2 &= ab + ac + \ldots \\ \&c. &= \&c. \\ S_n &= abc\ldots p. \end{aligned}$$

Then will

$$\begin{aligned} &(x+a)(x+b)(x+c)\ldots(x+p)(x+q) \\ &= (x+q)\{x^n + S_1 x^{n-1} + S_2 x^{n-2} + \ldots + S_n\} \\ &= x^{n+1} + S_1 x^n + S_2 x^{n-1} + \ldots + S_n x \\ &\qquad + q x^n + q S_1 x^{n-1} + \ldots + q S_n \\ &= x^{n+1} + S_1' x^n + S_2' x^{n-1} + \ldots + S_n' \text{ suppose,} \\ &\text{where } S_1' = S_1 + q = a + b + c + \ldots + p + q, \\ &\qquad S_2' = S_2 + q S_1 = ab + ac + \ldots + qa + qb + \ldots \\ &\qquad \&c. = \&c. \\ &\qquad S_n' = q S_n = abc\ldots pq. \end{aligned}$$

Hence it appears, that *if* the assumed law be true for $n$ factors, it will be true for $n+1$; but it *is* true for *three*; $\therefore$ for *four*; $\therefore$ &c. $\therefore$ generally true.

Now let $a = b = c = \ldots = p$.

Then $S_1 = a + a + a + \ldots$ to $n$ terms $= na$,

$S_2 = a^2 + a^2 + \ldots$ to as many terms as there are combinations of $n$ things taken *two* together

$$= \frac{n(n-1)}{1 \cdot 2} a^2, \text{ (Art. 114)}$$

$$\&c. = \&c.$$

$$S_n = a^n;$$

and $(x+a)(x+b)(x+c)\ldots(x+p)$ becomes $= (x+a)^n$;

$$\therefore (x+a)^n = x^n + n x^{n-1} a + \frac{n(n-1)}{1 \cdot 2} x^{n-2} a^2 + \ldots + a^n.$$

The general term being $\frac{n(n-1)\ldots(n-r+1)}{1.2\ldots\ldots\ldots\ldots r} x^{n-r}a^r$, and the number of terms $n+1$.*

COR. 1. $(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \ldots + x^n$.

COR. 2. It appears from the proposition proved in Art. 115, that the coefficients of terms at the same distance from the beginning and end of the series are the same; which is also otherwise apparent from the fact that $(x+a)^n = (a+x)^n$.

Ex. 1. $(a+b)^4 = a^4 + 4a^3b + \frac{4.3}{1.2}a^2b^2 + \frac{4.3.2}{1.2.3}ab^3 + b^4$,

$$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

Ex. 2. Find the coefficient of $a^5b^3$ in the expansion of $(a+b)^8$.

$$\text{The coefficient} = \frac{8.7.6}{1.2.3} = 56.$$

Ex. 3. Find the middle term of the expansion of $(1+x)^6$.

The middle term is the fourth;

$$\therefore \text{the term required} = \frac{6.5.4}{1.2.3}x^3 = 20\,x^3.$$

Ex. 4. $(2a-3x)^3$

$$= 2^3a^3 - 3.2^2.3a^2x + \frac{3.2}{1.2}2.3^2ax^2 - \frac{3.2.1}{1.2.3}3^3x^3$$

$$= 8a^3 - 36a^2x + 54ax^2 - 27x^3.$$

* The following may perhaps assist the student in comprehending this proof.

A little consideration will make it appear, that in multiplying together $n$ factors, $x+a_1$, $x+a_2 \ldots x+a_n$, the result must be such that each term shall be of $n$ dimensions, that is, the indices of the factors of each term added together shall be equal to $n$. What then will be the coefficient of the term involving $x^r$? It will consist of terms composed of $a_1\ a_2$ &c., such that the sum of the indices in each term shall be $n-r$; and since there is no reason why one combination of these letters should occur more than another, it will consist of *all* terms satisfying the condition of the sum of the indices being $n-r$.

Now let us suppose that $a_1 = a_2 = \&c. = a_n = a$; then each of the terms just mentioned will become $a^{n-r}$, and the number of them will be $\{nC(n-r)\}$, since it is obvious that this is the number of ways in which we can form out of $n$ letters combinations of the kind described. Hence in $(x+a)^n$ the term involving $x^r$ is $\{nC(n-r)\}\,x^r a^{n-r}$ or $\{nCr\}\,x^r a^{n-r}$: and this result is the Binomial Theorem for a positive integral index.

The Theorem may be applied to the expansion of a trinomial, as in the following examples.

Ex. 5. $(1 - x + x^2)^3$

$$\begin{aligned}
&= (1-x)^3 + 3(1-x)^2x^2 + 3(1-x)x^4 + x^6 \\
&= 1 - 3x + 3x^2 - x^3 \\
&\qquad\quad + 3x^2 - 6x^3 + 3x^4 \\
&\qquad\qquad\qquad + 3x^4 - 3x^5 + x^6 \\
&= 1 - 3x + 6x^2 - 7x^3 + 6x^4 - 3x^5 + x^6.
\end{aligned}$$

Ex. 6. $(1 + x + x^2)^n$

$$= 1 + n(x + x^2) + \frac{n(n-1)}{1\,.\,2}(x + x^2)^2 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}(x + x^2)^3 + \dots$$

$$\begin{aligned}
= 1 + nx + nx^2 &+ \frac{n(n-1)}{1\,.\,2}(x^2 + 2x^3 + x^4) \\
&+ \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}(x^3 + 3x^4 + 3x^5 + x^6) \\
&+ \dots\dots
\end{aligned}$$

$$\begin{aligned}
= 1 + nx + \left\{n + \frac{n(n-1)}{1\,.\,2}\right\}x^2 + \left\{n(n-1) + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\right\}x^3 \\
+ \dots\dots
\end{aligned}$$

119. The series, which has been proved for $(1 + x)^n$ when $n$ is positive and integral, may be shewn to be the true series in the case of $n$ being fractional or negative*; it is to be remarked, however, that there is this important distinction between the two cases, that when $n$ is a positive integer the series comes to an end, but when $n$ is fractional or negative the series will be infinitely extended. In fact we have seen that the general term of the series, when $n$ is a positive integer, is

* Strictly speaking, $(1+x)^n$ cannot be said to be expansible in a series proceeding by powers of $x$ unless the series be *convergent*; on this subject see Art. 122.

$$\frac{n(n-1)\ldots\ldots\ldots(n-r+1)}{1.2\ldots\ldots\ldots r}x^r;$$

and this becomes zero when

$$r = n + 1;$$

but if we prove that the same is the general term when $n$ is fractional or negative, it will be apparent that it never can become zero, and therefore the series can never terminate. We may, however, consider the series indefinitely continued even in the case of $n$ being a positive integer, only that after a certain number the terms will be in reality evanescent.

Before proceeding to the general consideration of the Binomial Theorem in the case of a fractional or negative index we will take a few particular examples, which will be useful by way of introduction.

The only series continued to an indefinite number of terms, with which we have hitherto become acquainted, is the geometrical series having the common ratio of the terms less than unity, (p. 71).

We saw in that case that

$$\frac{1}{1-r} = 1 + r + r^2 + \ldots\ldots\ldots(1).$$

Now $\frac{1}{1-r} = (1-r)^{-1}$; and the fact of $(1-r)^{-1}$ being capable of being expanded in a series such as (1), may naturally suggest to us the inquiry, whether that series may not be that which would result from supposing the Binomial Theorem, which has been proved for the case of $(1-r)^n$, $n$ being a positive integer, to be also true when $n$ is a negative integer. We have then

$$(1-r)^n = 1 - nr + \frac{n(n-1)}{1.2}r^2 - \ldots$$

when $r$ is a positive integer; write $-1$ in the place of $n$, and we have

$$(1-r)^{-1} = 1-(-1)r+\frac{(-1)(-1-1)}{1\,.\,2}r^2 - \ldots$$

$$= 1+r+r^2+\ldots\ldots\ldots\ldots\ldots(2).$$

The series (1) and (2) agree; hence we obtain the remarkable result, that $(1-r)^{-1}$ may be expanded in a series proceeding by powers of $r$ by supposing the Binomial Theorem to extend to that case.

Next let us take an example in which the index is fractional. Applying the ordinary method of extracting the square root to the quantity $1+x$, we have the subjoined operation;

$$\begin{array}{rl}
 & 1+x \ \left(1+\frac{x}{2}-\frac{x^2}{8}\ldots\right. \\
 & 1 \\
\hline
2+\frac{x}{2}\Big) & x \\
 & x+\frac{x^2}{4} \\
\hline
2+x-\frac{x^2}{8}\Big) & -\frac{x^2}{4} \\
 & -\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{64} \\
\hline
 & \frac{x^3}{8}-\frac{x^4}{64}
\end{array}$$

Hence we have

$$(1+x)^{\frac{1}{2}} = 1+\frac{x}{2}-\frac{x^2}{8}+\ldots\ldots(1).$$

Now let us see what will result from the supposition that the Binomial Theorem is true when the index is fractional: we have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \dots$$

in this series write $\frac{1}{2}$ in the place of $n$, and there results

$$(1+x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{1 \cdot 2}x^2 + \dots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \dots\dots\dots(2).$$

The series (1) and (2) agree; hence the supposition that the Binomial Theorem is true in this case leads to the same result as the application of the ordinary method of extracting the root.

We might take other examples, but probably the two which have been given will be sufficient as an introduction to the general treatment of the Binomial Theorem in the cases of negative and fractional indices. The comparison of a considerable number of examples would render it highly probable that the theorem was generally true: we now proceed to the proof that such is actually the case.

120. *To extend the Binomial Theorem to the case of fractional and negative indices.*

Let the series

$$1 + mx + \frac{m(m-1)}{1 \cdot 2}x^2 + \text{\&c.},$$

(*where m may be any quantity whatever*) be represented for shortness' sake by the symbol $f(m)$.

Then, according to the same notation,

$$f(n) = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \dots\dots\dots$$

Our first step will be to determine the form of the product $f(m) \times f(n)$: to do this we observe, that by actual multiplication it is clear that $f(m) \times f(n)$ will be a series proceeding by ascending powers of $x$,

$$= 1 + Ax + Bx^2 + \dots\dots\dots \text{ suppose.}$$

It would be possible to determine the coefficients $A$, $B$... by actual multiplication*; but we obtain them more simply by this consideration, that although the *values* of $A$, $B$,...... are altered by altering $m$ and $n$, yet their *forms*, that is, the manner in which $m$ and $n$ are involved in them, are the same whatever $m$ and $n$ may be; and therefore if we discover the form of the product $f(m) \times f(n)$ in the case of $m$ and $n$ being positive integers, we shall know its form whatever $m$ and $n$ may be.

But in that case $f(m) = (1 + x)^m$,

$$\text{and } f(n) = (1 + x)^n;$$

$$\therefore f(m) \times f(n) = (1 + x)^{m+n}$$

$$= f(m + n) \text{ by the notation;}$$

and hence, by the preceding reasoning, we must have *universally*

$$f(m) \times f(n) = f(m + n);$$

and in like manner we shall have for any number of factors,

$$f(m) \times f(n) \times f(p) \times \ldots\ldots\ldots = f(m + n + p + \ldots\ldots\ldots).$$

This being premised, in the formula

$$f(m) \times f(n) \times f(p) \ldots\ldots\ldots = f(m + n + p + \ldots\ldots\ldots),$$

$$\text{make } m = n = p = \ldots\ldots\ldots = \frac{\mu}{\nu},$$

where $\mu$ and $\nu$ are positive whole numbers, and let there be $\nu$ factors; then we have

$$\left\{f\left(\frac{\mu}{\nu}\right)\right\}^\nu = f(\mu)$$

$$= (1 + x)^\mu \text{ since } \mu \text{ is a positive integer;}$$

* In order to obtain the required result by this means, it would be necessary to obtain a few terms by actual multiplication, then to assume the form of the coefficient of $x^r$ and to prove that the form so assumed, if true for $x^r$, will be true for $x^{r+1}$.

$$\therefore (1+x)^{\frac{\mu}{\nu}} = f\left(\frac{\mu}{\nu}\right) = 1 + \frac{\mu}{\nu}x + \frac{\frac{\mu}{\nu}\left(\frac{\mu}{\nu}-1\right)}{1.2}x^2 + \ldots$$

which proves the theorem for fractional indices.

Again, in the formula

$$f(m) \times f(n) = f(m+n),$$

let $m = -n$, and let $n$ be positive;

$$\therefore f(-n) \times f(n) = f(0) = 1,$$

(since the series $1 + nx + \frac{n(n-1)}{1.2}x^2 + \ldots$ becomes 1 when $n = 0$;)

$$\therefore f(-n) = \frac{1}{f(n)} = \frac{1}{(1+x)^n} = (1+x)^{-n};$$

$$\therefore (1+x)^{-n} = f(-n) = 1 - nx + \frac{-n(-n-1)}{1.2}x^2 - \&c.$$

which proves the theorem for negative indices*.

120 (bis). As the proof given in the preceding article is of a very subtle kind, and perhaps likely to create some amount of perplexity, it may be as well to subjoin another method of treating the Binomial Theorem for the fractional and negative index. The method which we are about to give depends upon the following Lemma.

* The Binomial Theorem, as applied to the expansion of $(1+x)^n$, whether $n$ be positive or negative, integral or fractional, is due to Sir I. Newton. A method of raising a binomial to a positive power without going through the process of actual multiplication was known as early as the beginning of the sixteenth century, but this method only extended to finding the successive powers by means of tables calculated for the purpose, so that in order to raise a binomial to any power all the inferior powers had to be found. The next step was the method of raising a binomial to a positive integral power, without the intervention of the inferior powers, and this was the point at which the theorem had arrived before the time of Newton. He completed the theorem by observing that the same rule which served to expand a binomial raised to a positive power, would also serve to express the root or the reciprocal of a binomial in the form of an infinite series.

It may be well to remark, that the proof given in the text is not that which was given by the discoverer; in fact, Newton did not himself give any complete proof of his theorem, but assured himself of its truth by an inductive process. Many have been given since his time; that in the text is one of those given by Euler, and appears to be the most elegant which has been proposed; but the reasoning is somewhat subtle, and will require careful consideration on the part of the student who wishes to master it.

If $$A + Bx + Cx^2 + Dx^3 + \ldots = 0,$$
for all values of $x$ whatever, then we must have
$$A = 0,\ B = 0,\ C = 0,\ D = 0\ldots$$
The truth of this Lemma is seen at once if we consider that inasmuch as
$$-A = Bx + Cx^2 + Dx^3, \ldots$$
we shall have the value of $A$, which is fixed and unalterable, given in terms of $x$, which may have any value whatever, unless the above equations be true. The same thing may be expressed thus; if the equation
$$A + Bx + Cx^2 + \ldots = a + bx + cx^2 + \ldots,$$
be true whatever be the value of $x$, then
$$A = a,\ B = b,\ C = c, \ldots$$
This principle is one of great use in expanding algebraical quantities in the form of infinite series, and before proceeding to the application of it which we have immediately in hand, we will illustrate its utility by applying it to one or two particular examples.

Let it be required to expand $\dfrac{1}{1+x}$ in a series proceeding by powers of $x$.

Assume $$\frac{1}{1+x} = A + Bx + Cx^2 + Dx^3 + \ldots,$$
then multiplying by $(1 + x)$,
$$\begin{aligned} 1 = A &+ Bx + Cx^2 + Dx^3 + \ldots, \\ &+ Ax + Bx^2 + Cx^3 + \ldots; \end{aligned}$$
or $A - 1 + (A + B)\,x + (B + C)\,x^2 + (C + D)\,x^3 + \ldots = 0.$

Hence according to our principle,
$$\begin{aligned} A - 1 &= 0, \\ A + B &= 0, \\ B + C &= 0, \\ C + D &= 0; \end{aligned}$$
$$\therefore A = 1, \qquad B = -A = -1,$$
$$C = -B = +1, \qquad D = -C = -1;$$

$$\therefore \frac{1}{1+x} = 1 - x + x^2 - x^3 + \&c.$$

Again, let it be required to expand $\frac{1+x}{(1-x)^2}$ in a series proceeding by powers of $x$.

Assume $\quad \frac{1+x}{(1-x)^2} = A + Bx + Cx^2 + \dots;$

$$\begin{aligned}\therefore 1 + x &= (1 - 2x + x^2)(A + Bx + Cx^2 + \dots) \\ &= A + Bx + Cx^2 + \dots \\ &\qquad - 2Ax - 2Bx^2 - \dots \\ &\qquad\qquad + Ax^2 + \dots \\ &= A + (B - 2A)x + (C - 2B + A)x^2 + \&c.\end{aligned}$$

Hence according to our principle,

$$\begin{aligned}&A = 1, \\ &B - 2A = 1, \text{ or } B = 2A + 1 = 3, \\ &C - 2B + A = 0, \text{ or } C = 2B - A = 6 - 1 = 5;\end{aligned}$$

$$\therefore \frac{1+x}{(1-x)^2} = 1 + 3x + 5x^2 + \&c.$$

These examples will sufficiently illustrate the method which is called that of Indeterminate Coefficients, and which we will now apply to the expansion of $(a + x)^n$ when $n$ is a fractional or negative quantity.

Let us assume that

$$(1 + x)^n = A + A_1 x + A_2 x^2 + \dots + A_r x^r + \dots,$$

then in the first place, since this series is true whatever value we assign to $x$, it will be true when $x = 0$; let $x = 0$, then we have

$$1 = A,$$

$$\text{and } \therefore (1 + x)^n = 1 + A_1 x + A_2 x^2 + \dots + A_r x^r + \dots (1),$$

and therefore also,

$$\begin{aligned}(a + x)^n = a^n \left(1 + \frac{x}{a}\right)^n &= a^n \left\{1 + A_1 \frac{x}{a} + A_2 \frac{x^2}{a^2} + \dots + A_r \frac{x^r}{a^r} + \dots\right\} \\ &= a^n + A_1 a^{n-1} x + A_2 a^{n-2} x^2 + \dots + A_r a^{n-r} x^r + \dots (2).\end{aligned}$$

We will next, by means of the principle above explained, determine the relation which subsists between the coefficients of successive powers of $x$. For this purpose write $y + z$ for $x$ in (1), and we have

$$(1 + y + z)^n = 1 + A_1 (y + z) + A_2 (y + z)^2 + \ldots + A_r (y + z)^r + \ldots$$
$$= 1 + A_1 y + A_2 y^2 + \ldots + A_r y^r + \ldots$$
$$+ z (A_1 + 2A_2 y + \ldots + rA_r y^{r-1} + \ldots)$$
$$+ \ldots\ldots\ldots$$

Again, write $1 + y$ instead of $a$, and $z$ instead of $x$ in (2), and we have

$$(1 + y + z)^n = (1 + y)^n + A_1 (1 + y)^{n-1} z + A_2 (1 + y)^{n-2} z^2 + \ldots.$$

By our general principle the coefficients of $z$ in these two expressions for $(1 + y + z)^n$ must be equal, and therefore

$$A_1 (1 + y)^{n-1} = A_1 + 2A_2 y + \ldots + rA_r y^{r-1} + \ldots;$$
$$\therefore A_1 (1 + y)^n = (1 + y) (A_1 + 2A_2 y + \ldots + rA_r y^{r-1} + \ldots)$$
$$\text{or } A_1 (1 + A_1 y + A_2 y^2 + \ldots)$$
$$= A_1 + (2A_2 + A_1) y + \ldots + (rA_r + \overline{r - 1}\, A_{r-1}) y^{r-1} + \ldots$$

Comparing the coefficients of $y^{r-1}$ on the two sides of this equation, there results

$$A_1 A_{r-1} = rA_r + (r - 1) A_{r-1},$$
$$\therefore A_r = \frac{A_1 - r + 1}{r} A_{r-1}.$$

We have here the general relation between each coefficient in the series and that which precedes it.

$$\text{Let } r = 2, \therefore A_2 = \frac{A_1 - 1}{2} A_1,$$
$$r = 3, \therefore A_3 = \frac{A_1 - 2}{3} A_2 = \frac{(A_1 - 2)(A_1 - 1) A_1}{3 \,.\, 2 \,.\, 1},$$

and so on. Hence

$$(1 + x)^n = 1 + A_1 x + \frac{A_1 (A_1 - 1)}{1 \,.\, 2} x^2 + \frac{A_1 (A_1 - 1)(A_1 - 2)}{1 \,.\, 2 \,.\, 3} x^3 + \ldots$$

It still remains to find $A_1$; and this we must do separately for the two cases of $n$ being fractional and $n$ being negative.

First, let $n$ be fractional and equal to $\frac{\mu}{\nu}$.

Then $(1+x)^n = (1+x)^{\frac{\mu}{\nu}} = 1 + A_1 x + \dots$

$$\therefore (1+x)^\mu = (1 + A_1 x + \dots)^\nu,$$

or, applying the Binomial Theorem as proved for a positive integral index,

$$1 + \mu x + \dots = 1 + \nu A_1 x + \dots;$$

$$\therefore \mu = \nu A_1,$$

$$\text{or } A_1 = \frac{\mu}{\nu} = n.$$

Secondly, let $n$ be negative and equal to $-m$.

Then $(1+x)^n = (1+x)^{-m} = \dfrac{1}{(1+x)^m} = \dfrac{1}{1 + mx + \&c.}$

$= 1 - mx + \dots$ by actual division,

$= 1 + nx + \dots$

Hence whether $n$ be fractional or negative, $A_1 = n$, and therefore

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \dots$$

We have thus established by a different method from that given in the preceding Article the same conclusion; the student will find this second investigation worthy of study, not as a substitute for but as a supplement to the former, supplying him with a new point of view from which to regard a rather difficult theorem. It may be remarked that the method of proof depending upon the principle of indeterminate coefficients may be varied in its details; it is usual, for instance, to prove that $A_1 = n$ before the relation of $A_2$ to $A_1$, of $A_3$ to $A_2$, &c., is investigated; the opposite course has however here been followed, because by doing so we exhibit more clearly the amount of assistance which is yielded by the method of indeterminate coefficients; we see that that method suffices to give us the relation of successive coefficients, but that it leaves entirely undetermined the coefficient of the first power of $x$.

We now proceed to illustrate the Theorem in the case of fractional and negative indices by a few examples.

Ex. 1.

$$(1+x)^{-2} = 1 - 2x + \frac{-2(-2-1)}{1\,.\,2}x^2 + \frac{-2(-2-1)(-2-2)}{1\,.\,2\,.\,3}x^3 + \&c.$$

$$= 1 - 2x + \frac{2\,.\,3}{1\,.\,2}x^2 - \frac{2\,.\,3\,.\,4}{1\,.\,2\,.\,3}x^3 + \&c.$$

$$= 1 - 2x + 3x^2 - 4x^3 + \&c.$$

Ex. 2. $$(a+bx)^{-1} = \frac{1}{a}\left(1 + \frac{bx}{a}\right)^{-1}$$

$$= \frac{1}{a}\left\{1 - \frac{bx}{a} + \frac{-1(-1-1)}{1\,.\,2}\frac{b^2x^2}{a^2} + \frac{-1(-1-1)(-1-2)}{1\,.\,2\,.\,3}\frac{b^3x^3}{a^3} + \ldots\right\}$$

$$= \frac{1}{a}\left\{1 - \frac{bx}{a} + \frac{1\,.\,2}{1\,.\,2}\frac{b^2x^2}{a^2} - \frac{1\,.\,2\,.\,3}{1\,.\,2\,.\,3}\frac{b^3x^3}{a^3} + \ldots\right\}$$

$$= \frac{1}{a}\left(1 - \frac{bx}{a} + \frac{b^2x^2}{a^2} - \frac{b^3x^3}{a^3} + \ldots\ldots\right),$$

which is a result obtainable by actual division; it may be observed, however, that if the result had been so obtained we should have had also a *remainder*, of which there is no trace in the preceding series: the fact is, that the series obtained by the Binomial Theorem can only be considered as numerically equal to the quantities expanded, when the series are *convergent*. When the convergence is very rapid the Binomial Theorem may be conveniently used for purposes of approximation, as will be seen in the next Example.

Ex. 3. To find an approximate value of $\sqrt[p]{N^p + z}$, when $z$ is much smaller than $N^p$.

$$(N^p + z)^{\frac{1}{p}} = N\left(1 + \frac{z}{N^p}\right)^{\frac{1}{p}}$$

$$= N\left\{1 + \frac{1}{p}\frac{z}{N^p} + \frac{\frac{1}{p}\left(\frac{1}{p} - 1\right)}{1\,.\,2}\frac{z^2}{N^{2p}} + \ldots\ldots\right\}.$$

By taking a few terms of this series we may obtain the result with a considerable degree of accuracy.

Ex. 4. To find an approximate value of $\sqrt{50}$.

$$\sqrt{50} = (49+1)^{\frac{1}{2}} = 7\left(1+\frac{1}{49}\right)^{\frac{1}{2}}$$

$$= 7\left\{1+\frac{1}{2}\frac{1}{49}+\frac{\frac{1}{2}(\frac{1}{2}-1)}{1\,.\,2}\frac{1}{49^2}+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1\,.\,2\,.\,3}\frac{1}{49^3}+\ldots\right\}$$

$$= 7\left(1+\frac{1}{2}\frac{1}{49}-\frac{1}{8}\frac{1}{49^2}+\frac{1}{16}\frac{1}{49^3}-\ldots\right)$$

$$= 7\left(1+\frac{.020408}{2}-\frac{.000416}{8}+\frac{.000008}{16}-\ldots\right)$$

$$= 7\,(1.010204 - .000052 + \ldots)$$

$$= 7\,(1.010152) = 7.071064,$$

which is correct to five places of decimals.

Ex. 5. The following example is given on account of its bearing upon the theory of logarithms. (See page 102, Note.)

$$(1+mx)^{\frac{1}{m}} = 1+\frac{1}{m}mx+\frac{\frac{1}{m}\left(\frac{1}{m}-1\right)}{1\,.\,2}m^2x^2+\frac{1}{m}\frac{\left(\frac{1}{m}-1\right)\left(\frac{1}{m}-2\right)}{1\,.\,2\,.\,3}m^3x^3+\ldots$$

$$= 1+x+\frac{1-m}{1\,.\,2}x^2+\frac{(1-m)(1-2m)}{1\,.\,2\,.\,3}x^3+\ldots$$

Let $x = 1$,

$$\therefore\ (1+m)^{\frac{1}{m}} = 1+1+\frac{1-m}{1\,.\,2}+\frac{(1-m)(1-2m)}{1\,.\,2\,.\,3}+\ldots$$

If we suppose $m$ to be indefinitely small, the second member of this equation becomes equal to the series

$$1+1+\frac{1}{1\,.\,2}+\frac{1}{1\,.\,2\,.\,3}+\ldots;$$

the value of this series may be easily calculated and will be found to be equal to 2.7182818 approximately. This is a number of continual occurrence in mathematics, and is usually denoted by the letter $e$; we have therefore

$$e = \text{the value of } (1+m)^{\frac{1}{m}} \text{ when } m = 0.$$

Now make $m = nx$,

$$\therefore\ e = \text{the value of } (1 + nx)^{\frac{1}{nx}} \text{ when } m = 0, \text{ or when } n = 0;$$

$$\therefore\ e^x = \text{the value of } (1 + nx)^{\frac{1}{n}} \text{ when } n = 0.$$

$$\text{But } (1 + nx)^{\frac{1}{n}} = 1 + x + \frac{1 - n}{1\,.\,2}x^2 + \frac{(1 - n)(1 - 2n)}{1\,.\,2\,.\,3}x^3 + \ldots$$

$$= 1 + x + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \ldots \text{ when } n = 0.$$

$$\therefore\ e^x = 1 + x + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \ldots$$

We may in like manner express $a^x$, where $a$ is any number whatever, in a series proceeding by powers of $x$. For let $L$ be such a quantity that

$$a = e^L;$$

$$\text{then } a^x = e^{Lx} = 1 + Lx + \frac{L^2x^2}{1\,.\,2} + \frac{L^3x^3}{1\,.\,2\,.\,3} + \ldots$$

The explanation of the method of finding the quantity $L$ would carry us beyond our present purpose.

121. *To find the greatest term in the expansion of* $(1 + x)^n$.

The $(r + 1)^{\text{th}}$ term of the expansion is formed from the $r^{\text{th}}$ by multiplying it by the quantity $\dfrac{n - r + 1}{r}x$; so long therefore as $r$ is such as to make this quantity greater than 1 the terms will increase, but when it becomes less than 1 the terms will decrease. We have therefore to determine the value of $r$ which will make $\dfrac{n - r + 1}{r}x$ less than 1; but in doing so it must be observed, that it is the *numerical* value which must be less than 1, and therefore there will be two cases to consider, according as $\dfrac{n - r + 1}{r}$ is positive or negative.

First, let it be positive. Then

$$\frac{n-r+1}{r}x \text{ will be} < 1,$$

$$\text{provided } (n+1)x < r(1+x),$$

$$\text{or } r > (n+1)\frac{x}{1+x}.$$

If therefore we take $r$ equal to the whole number next greater than $(n+1)\dfrac{x}{1+x}$, the corresponding term will be the greatest.

Secondly, let $\dfrac{n-r+1}{r}$ be negative, then will $\dfrac{r-n-1}{r}$ be positive, and we shall have

$$\frac{r-n-1}{r}x < 1,$$

$$\text{provided } -(n+1)x < r(1-x),$$

$$\text{or } r > -(n+1)\frac{x}{1-x},$$

and the value of $r$ required will be the whole number next greater than $-(n+1)\dfrac{x}{1-x}$.

Let us now determine the circumstances under which each of these rules will apply.

1. Suppose $\dfrac{n-r+1}{r}x$ positive, then must

$$n+1 \text{ be} > r$$

$$> (n+1)\frac{x}{1+x} \text{ à fortiori,}$$

$$\text{or } n+1 > 0;$$

*i.e.* $n$ must be either positive or a negative quantity less than unity.

2. Suppose $\dfrac{n-r+1}{r}x$ negative, then must

$$n+1 \text{ be} < r$$

$$< -(n+1)\frac{x}{1-x}+1,$$

$$\left\{\text{since } r \text{ is the integer next greater than } -(n+1)\frac{x}{1-x}\right\},$$

$$\text{or } n+1 < 1-x,$$

$$n < -x,$$

or $n$ must be negative.

Ex. 1. To find the greatest term in the expansion of $\left(1+\frac{1}{4}\right)^8$.

In this case $n=8$, $x=\frac{1}{4}$, $\therefore (n+1)\frac{x}{1+x} = 9 \times \frac{1}{5} = \frac{9}{5}$, or the second term is the greatest.

Let us verify this result;

$$\left(1+\frac{1}{4}\right)^8 = 1 + \frac{8}{4} + \frac{8 \cdot 7}{1 \cdot 2} \cdot \frac{1}{4^2} + \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} \cdot \frac{1}{4^3} + \ldots\ldots$$

$$= 1 + 2 + \frac{7}{4} + \frac{7}{8} + \ldots\ldots$$

in which series the second is evidently the greatest term.

Ex. 2. To find the greatest term in the expansion of $\left(1+\frac{1}{5}\right)^{-12}$.

In this case $n = -12$, $x = \frac{1}{5}$;

$$\therefore -(n+1)\frac{x}{1-x} = (12-1)\frac{1}{4} = \frac{11}{4} = 2\frac{3}{4},$$

or the third term is the greatest.

To verify this result,

$$\left(1+\frac{1}{5}\right)^{-12} = 1 - \frac{12}{5} + \frac{12 \cdot 13}{1 \cdot 2}\frac{1}{5^2} - \frac{12 \cdot 13 \cdot 14}{1 \cdot 2 \cdot 3}\frac{1}{5^3} + \ldots\ldots$$

$$= 1 - \frac{12}{5} + \frac{78}{25} - \frac{364}{125} + \ldots\ldots$$

$$= 1 - \frac{300}{125} + \frac{390}{125} - \frac{364}{125} + \ldots\ldots$$

which verifies the result.

122. It does not always happen that the value of $(1+x)^n$ can be calculated approximately by taking a considerable number of terms of the series $1 + nx + \frac{n(n-1)}{1.2}x^2 + \ldots\ldots$; in other words it is not always true that this series is convergent. It will be desirable therefore to ascertain the condition of the convergency of the series, which may be done as follows.

Let $P$ represent the $p^{th}$ term, and $R$ the sum of the terms after the $p^{th}$; so that

$$R = P\left\{\frac{n-p+1}{p}x + \frac{(n-p+1)(n-p)}{p(p+1)}x^2 + \ldots\ldots\right\}$$

$$= P\left\{\left(\frac{n+1}{p} - 1\right)x + \left(\frac{n+1}{p} - 1\right)\left(\frac{n+1}{p+1} - 1\right)x^2 + \ldots\ldots\right\}$$

Now suppose $p$ to be taken very large, then the quantities $\frac{n+1}{p}, \frac{n+1}{p+1}, \ldots\ldots$ become very small, and we shall have approximately,

$$R = P\{-x + x^2 - x^3 + \ldots\ldots\},$$

a geometrical series of which the sum cannot be calculated unless $x$ be less than unity, (Art. 105). Consequently the condition of convergency for the binomial series is that $x$ shall be a proper fraction.

It may be noticed that the method of Art. 121 will sometimes assign a greatest term to the series although the condition of convergency may not be satisfied; in this case however it will be found that the terms decrease after the term so found and then again increase, and the method of the article referred to only determines the condition of the $(r+1)^{th}$

term being less than the $r^{\text{th}}$ without introducing the condition that there shall be no greater term afterwards.

For instance, in the series for $\left(1+\frac{5}{2}\right)^{\frac{2}{3}}$, we find that for the greatest term

$$r > \left(1+\frac{2}{5}\right)\frac{5}{5+2} > 1;$$

$$\therefore\ r = 2$$

$$\text{Now}\left(1+\frac{5}{2}\right)^{\frac{2}{3}} = 1+1+\frac{\frac{2}{3}(\frac{2}{3}-1)}{1.2}\left(\frac{5}{2}\right)^2+\frac{\frac{2}{3}(\frac{2}{3}-1)(\frac{2}{3}-2)}{1.2.3}\left(\frac{5}{2}\right)^3+\ldots$$

$$= 1+1-\frac{3}{4}+\frac{7}{8}-\&c.$$

and we perceive that the second term is greater than the third; but after the third the terms again increase, and there is in reality no greatest term.

## ON LOGARITHMS.

123. DEF. The logarithm of a number $N$ is the value of $x$ which satisfies the equation $a^x = N$, where $a$ is some given number.

Thus if $a$ be 10, the logarithm of 100 is 2, that of 1000 is 3; and that of any number between 100 and 1000 will be greater than 2 and less than 3, so that it may be represented by 2 followed by places of decimals.

If we give $a$ any value, as 10, it is possible to find the values of $x$ corresponding to all values of $N$, that is, to find the logarithms of all numbers to the *base* 10; suppose these found and registered in tables, these will be the common tables of logarithms; we shall see of what use they may be made from the following propositions.

123. *The logarithm of the product of two numbers is the sum, and the logarithm of the quotient is the difference of the logarithms of the numbers.*

For let $a^x = N$, $a^y = N'$, where $NN'$ are any two numbers, and $x\ y$ their logarithms to the base $a$,

$$\text{then } a^{x+y} = NN',$$

but by definition $x+y$ is the logarithm of $NN'$ to base $a$, or (as we usually write it) $x+y=\log_a NN'$;

$$\therefore \log_a NN' = \log_a N + \log_a N'.$$

In like manner

$$\frac{N}{N'} = a^{x-y},$$

$$\text{and } \therefore \log_a \frac{N}{N'} = \log_a N - \log_a N'.$$

125. *The logarithm of any power of a number is the logarithm of the number multiplied by the index which expresses the power.*

$$\begin{aligned} \text{Suppose} \qquad a^x &= N, \\ \text{then } a^{px} &= N^p, \\ \text{or } px &= \log_a N^p \text{ by definition}, \\ \text{but } x &= \log_a N; \\ \therefore \log_a N^p &= p \log_a N. \end{aligned}$$

In like manner

$$\log_a N^{\frac{1}{p}} = \frac{1}{p} \log_a N.$$

126. Hence it appears that by means of a table of logarithms, *multiplication* may be performed by addition, *division* by subtraction, *involution* by multiplication, and *evolution* by division.

For suppose that we possess such a table, and that we wish to multiply together two numbers $N$ and $N'$. We look for the logarithms of these two numbers, add them together, and then look among the logarithms for the sum thus found, the number corresponding to that logarithm will be $NN'$: and so of the other operations. From this it will be easily understood that the use of logarithms greatly facilitates long calculations.

127. *To explain the advantage of choosing* 10 *as the base of a system of logarithms**.

Suppose we have any two numbers in which the digits are the same, but which differ from each other in the position of the unit's place: for example, 137 and 13700. Then

$$13700 = 137 \times 100 = 137 \times 10^2;$$

$$\therefore \log_{10} 13700 = \log_{10} 137 + 2.$$

Hence the logarithms of the two numbers in question differ from each other only in this, that the larger one has 2 added to it, the decimal parts of the two being the same. And we may, in like manner, conclude that the decimal parts of the logarithms of all numbers having the same digits, but a different unit's place, are the same. Hence, if we have a rule for assigning the integral part of a logarithm, the tables need not contain the logarithms of all numbers, but only of those in which the digits are different.

The integral part of a logarithm is called the *characteristic*, the decimal part the *mantissa*.

128. *To find a rule for ascertaining the characteristic of the logarithm of any number.*

We have $\log_{10} 10^2 = 2$, and $\log_{10} 10^3 = 3$; hence the logarithm of any number between 100 and 1000, i.e. of any number

* Logarithms calculated to the base 10 are sometimes called Briggs' logarithms, from the name of their inventor. The first deviser of the method of logarithms however was not Briggs, but Napier, baron of Marcheston near Edinburgh, who, in 1614, published what he called *Canon mirabilis Logarithmorum*. The base of Napier's system is the value of the series

$$1 + 1 + \frac{1}{1 \, . \, 2} + \frac{1}{1 \, . \, 2 \, . \, 3} + \frac{1}{1 \, . \, 2 \, . \, 3 \, . \, 4} + \ldots\ldots$$

which may be easily shewn to be equal to 2.7182818........., and which in the actual calculation of logarithms and in many mathematical processes is the most convenient base; it is usually denoted by the letter $e$. (See p. 95, Ex. 5.)

Briggs' logarithms may be easily deduced from Napier's; for, if $N$ be any number, we have by definition,

$$N = 10^{\log_{10} N},$$

$$\therefore \text{ taking logarithms, } \log_e N = \log_{10} N \times \log_e 10,$$

$$\therefore \log_{10} N = \frac{1}{\log_e 10} \, . \, \log_e N.$$

Hence Briggs' logarithms may be deduced from Napier's by multiplying the latter by the quantity $\frac{1}{\log_e 10}$, which is a quantity easily calculated, and is called the *modulus*.

composed of three digits, is between 2 and 3, and therefore the characteristic is 2; similarly for numbers of four digits the characteristic is 3; and generally the characteristic is one less than the number of digits. If the number be decimal we must have a negative characteristic, for

$$\log_{10} 1 = 0,$$

$$\log_{10}\tfrac{1}{10} = \log_{10} .1 = -1,$$

$$\text{and } \log_{10}\tfrac{1}{100} = \log_{10} .01 = -2;$$

hence the logarithm of a number between 1 and .1 is less than 0 and greater than $-1$, and may therefore be represented by $-1 +$ a mantissa; in like manner the logarithm of a number between .1 and .01 will be $-2 +$ a mantissa; and, generally, the characteristic will be one greater than the number of cyphers which precede the first significant figure.*

129. *On the use of logarithmic tables.*

Since by the preceding article we know at once the characteristic of the logarithm of any proposed number, it is usual in tables to give only the mantissa of the logarithm, leaving the characteristic to be supplied by the calculator. Suppose, for instance, we required the logarithm of 3.7192; looking for the number 37192 in the tables, we find the figures 5704495; hence we conclude, that the logarithm required is .5704495. If the number had been 371.92, the logarithm would have been 2.5704495; and for .037192, the logarithm would be $\bar{2}.5704495$, or $-2 + .5704495$; and so in other cases.

Good logarithmic tables are usually calculated for 5 figures, but the logarithms of numbers of 6 figures may be found very simply if we assume this principle, that the difference between the logarithms of two numbers not differing much from each other is proportional to the difference of the numbers. For example, the mantissa of the number 365120 is 5624356, that of 365130 is 5624475, and the difference between these is 119;

* The principle of making the mantissæ always positive is evidently only conventional: the logarithm of a number between .1 and .01 (for example) might as easily be denoted by $-1 -$ a mantissa, as by $-2 +$ a mantissa.

now suppose we wish to find the mantissa of the number 365124, then since the difference for 10 is 119, we assume that the difference for 4 will be $\frac{4}{10} \times 119$, or 48 nearly.

Or more generally, we assume that

$$\log_{10}(N+n) - \log_{10} N = \frac{n}{10}\{\log_{10}(N+10) - \log_{10} N\}.$$

We say *assume*, but in a complete treatise on the subject the proposition would be not assumed, but proved.

To render the process of finding the logarithm of a number of 6 digits more easy, tables are supplied with auxiliary tables, called *tables of proportional parts*. These are simply the results of the formula $\frac{n}{10}\{\log(N+10) - \log_{10} N\}$ reduced to numbers for each value of $n$ from 1 up to 9. A separate table is not required for the difference between each pair of logarithms, because in looking through logarithmic tables it will be easily seen that the difference remains the same for a considerable number of logarithms. Thus in the example just now taken, the table of proportional parts is as in the margin, but this table serves for all numbers from 364850 up to 367830. These tables of proportional parts render the process of finding logarithms of numbers of 6 figures very easy, since we have only to inspect the table and ascertain at once the quantity to be added to the logarithm given in the table.

| 119 | |
|---|---|
| 1 | 12 |
| 2 | 24 |
| 3 | 36 |
| 4 | 48 |
| 5 | 60 |
| 6 | 71 |
| 7 | 83 |
| 8 | 95 |
| 9 | 107 |

130. The preceding articles contain so much of the theory of logarithms, as is necessary to render their utility obvious and the mode of using them intelligible; the actual calculation of them would involve us in series with which the student is not at present acquainted, and for which, if he be desirous of pursuing the subject, he is referred to other treatises.

# PLANE TRIGONOMETRY.

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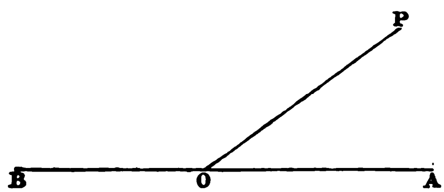
1. The science of *plane trigonometry*, according to the strict meaning of the words, treats of the *measurement* of *plane triangles*; we may however consider the name as applicable to the more general subject of the measurement of plane *angles*, of which the measurement of triangles forms an important part.

2. The term *angle* will be used in this subject in a more extended sense than that which is attached to it in Euclid's elements, for an angle according to Euclid's definition cannot exceed two right angles, and indeed, according to our ordinary conception of an *angle* or *corner*, it is manifest that there can be no such thing as an angle exceeding that limit: but there is no such restriction in Trigonometry; in that science the magnitude of an angle is unlimited. To make this understood, let $BOA$ be a fixed straight line, and $OP$ a line which revolves about $O$, and which at first coincided with $OA$. Then we say, that when $OP$ is in the position represented in the figure, it has described the angle $AOP$; but this mode of conceiving an angle admits of extension to angles of any magnitude, for we may suppose $OP$ to revolve beyond $OB$ and so describe an angle greater than two right angles, or more generally, we may suppose it to describe an angle of any magnitude whatever.




3. The same thing may be put in a slightly different point of view, by considering the point $P$ to trace out a circle with centre $O$. Then it is proved by Euclid, (VI. 33), that in the same circle the angle standing on any arc is proportional

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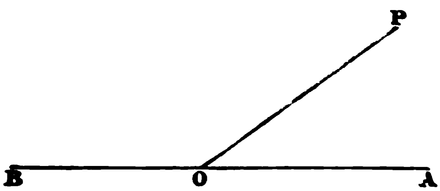



3. The same thing may be put in a slightly different point of view, by considering the point *P* to trace out a circle with centre *O*. Then it is proved by Euclid, (VI. 33), that in the same circle the angle standing on any arc is proportional

In this case, $F = 27.1544$

$$\frac{F}{10} = 2.71544$$

$$\therefore F - \frac{F}{10} = 24.43896$$

$$\begin{array}{r} 60 \\ \hline 26.33760 \\ 60 \\ \hline 20.25600 \end{array}$$

$$\therefore E = 24^{\circ}\ 26'\ 20''\ .256$$

Ex. 2. To find how many grades, minutes, &c., are contained in the angle $23^{\circ}\ 17'\ 51''$.

Here we must reduce $17'\ 51''$ to the decimal of a degree.

$$\begin{array}{rl} 60) & 51 \\ \hline 60) & 17.85 \\ \hline & .2975 \end{array}$$

$$\therefore E = 23.2975$$

$$\frac{E}{9} = 2.58861\dot{1}$$

$$\therefore E + \frac{E}{9} = 25.88611\dot{1}$$

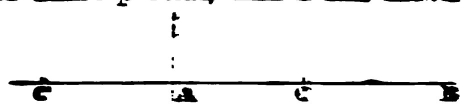
$$\therefore F = 25^{g}\ 88'\ 61''\ 11'''.........$$

Another mode of measuring angles will be given hereafter. (See Art. 52).

## ON THE USE OF THE SIGNS + AND − TO INDICATE THE DIRECTIONS OF LINES.

6. The primary use of the signs + and − is, as we have seen (Algebra, Art. 5), to denote addition and subtraction; nevertheless, we found that these signs immediately introduced the notion of *negative* quantities, and we illustrated the meaning of a negative quantity by a debt, which may be looked upon as a quantity *to be* subtracted.

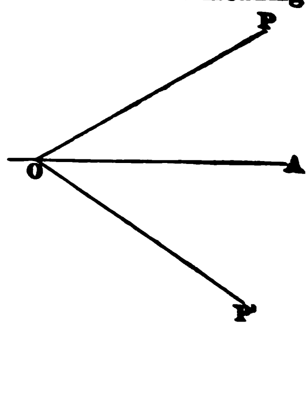
We must now still further generalize the meaning of the signs + and −, and it will be easy to shew that if a line drawn in one direction be called *positive*, then a line drawn in the opposite direction will properly be accounted negative. To make this appear, let $AB$ be any straight line




of length $a$, and let us cut off from it a portion $CB = b$, then the remainder $AC = a - b$. Now so long as $b$ is less than $a$, $C$ lies to the right of $A$ and $a - b$ is positive; but suppose we endeavour to cut off from $AB$ a portion $BC'$ greater than itself, then (although the operation is really impossible, yet) according to the analogy of the preceding operation we shall have for the remainder $AC'$, which is a line drawn to the *left* of $A$ and is represented by $-(b-a)$, or by a *negative* quantity. Hence then we see how that if a line drawn in one direction is accounted positive, then a line drawn in the opposite direction is properly accounted negative.

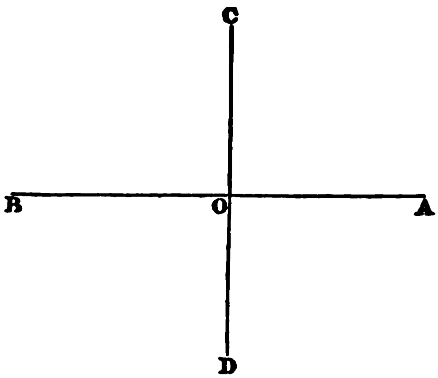
7. A more comprehensive view of this subject may be given, by saying that the signs + and − denote any exactly opposite qualities of a quantity; we are at liberty to take these signs in such a sense, because it includes the original meaning, viz. that of addition and subtraction, for *additive* and *subtractive* are qualities exactly the reverse of each other. And if we take the signs + and − as having this meaning, we shall see at once, that among other qualities, they properly designate opposition in *direction* when applied to lines.

As an example, let us consider what will be the meaning of a negative *angle*. Suppose the line $OP$ by its revolution about $O$ and *upwards* from $OA$ to describe the angle $AOP$, and let angles described in this manner be considered *positive*. Then if the line revolve *downwards* from $OA$ and describe the angle $AOP'$, this angle will be properly accounted negative.



The same explanation would apply to negative *arcs*.

8. In what follows, we shall consider that if $AOB$, $COD$, are two lines at right angles to each other, then lines drawn parallel to $AOB$ are positive if to the right, negative if to the left of $CD$, and lines drawn parallel to $COD$ are positive if drawn above $AOB$, and negative if below it.




We shall see immediately the great advantage of the preceding conventions.

## ON THE TRIGONOMETRICAL FUNCTIONS OF AN ANGLE.

Fig. I.

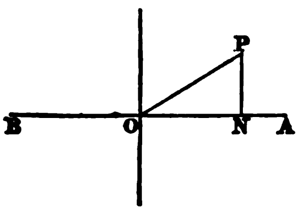



Fig. II.

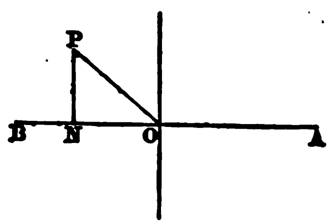



Fig. III.

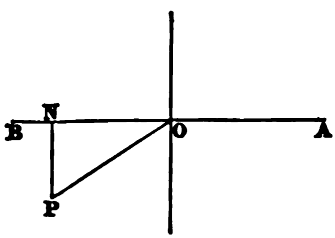
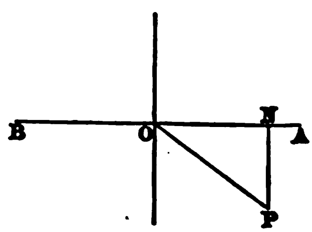



Fig. IV.




9. Let the line $OP$ revolving from the initial position $OA$ about $O$ describe the angle $AOP$: from $P$ let fall the perpendicular $PN$ upon the line $AOB$, then the ratio

(1) $\dfrac{PN}{OP}$ is defined to be the *sine* of the angle $AOP$.

(2) $\dfrac{ON}{OP}$ ........................ *cosine* ....................

(3) $\dfrac{PN}{ON}$ ........................ *tangent* ....................

(4) $\dfrac{ON}{PN}$ ........................ *cotangent* ....................

(5) $\dfrac{OP}{ON}$ ........................ *secant* ....................

(6) $\dfrac{OP}{PN}$ ........................ *cosecant* ....................

(7) $1-\dfrac{ON}{OP}$ ........................ *versedsine* ....................

The ratios to which we have just assigned names are called the trigonometrical *functions* of the angle, that is, quantities which depend upon that angle for their value, and (conversely) which being given determine the value of the angle.

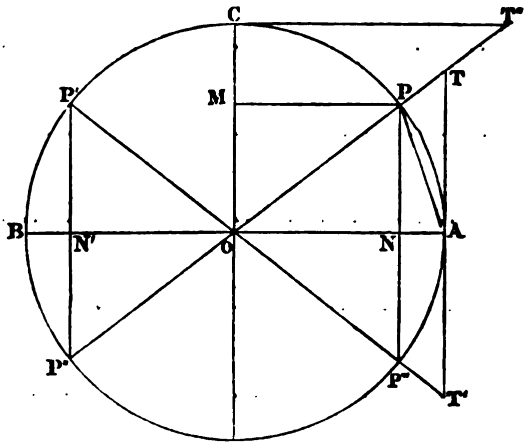
For shortness' sake we usually denote an angle by some single letter, as for instance $A$; and we write the names of the functions above defined thus, $\sin A$, $\cos A$, $\tan A$, $\cot A$, $\sec A$, $\operatorname{cosec} A$, $\operatorname{vers} A$*.

10. The meaning of these names†, and the relations of

* It is sometimes convenient to denote by a single symbol the angle whose sine is any given quantity; the following is the notation usually adopted; $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, &c. are symbols taken to represent respectively the angle whose sine is $x$, the angle whose cosine is $x$, the angle whose tangent is $x$, &c. The reasons for choosing this notation it is not necessary to discuss here.

† In the older treatises on Trigonometry it was usual to define the line $PN$ to be the sine of the angle $PON$ or of the arc $AP$, $ON$ to be the cosine, and so on. The disadvantage of this method is that in order to make an angle determinable from its sine, it is necessary to state what is the radius of the circle in which the lines are drawn. If however we suppose the radius of the circle to be *unity* this method of definition will coincide with that given in the text.

the trigonometrical functions to each other, will be seen more distinctly from another mode of defining them.




Suppose the point $P$ at the extremity of the revolving line $OP$ to trace out a circle of radius $r$. At the point $A$, which is the initial position of $P$, draw the tangent $TAT'$, and let $OP$ be produced to meet $AT$ in $T$; also draw $PN$ perpendicular to $OA$; then we may define the trigonometrical function of the angle $POA$ or $A$, thus:

$$\sin A = \frac{PN}{r},$$

$$\tan A = \frac{AT}{r}\left(= \frac{PN}{ON}, \text{ by similar triangles}\right),$$

$$\sec A = \frac{OT}{r}\left(= \frac{OP}{ON}, \text{ by similar triangles}\right),$$

$$\text{vers } A = \frac{AN}{r}.$$

With regard to the three other functions we may observe that the *complement* of an angle is its defect of a right angle, and that *cosine* merely signifies sine of the *complement*, *cotangent* tangent of the *complement*, and *cosecant* secant of the *complement*. Hence these functions need no new definition;

but if we take $AOC$ a right angle and draw the tangent $CT'$, and $PM$ perpendicular to $OC$, and produce $OP$ to meet $CT'$ in $T'$ we shall have,

$$\cos A = \sin COP = \frac{PN}{r} = \frac{ON}{OP}.$$

$$\cot A = \tan COP = \frac{CT'}{r} = \frac{PM}{OM} = \frac{ON}{PN}.$$

$$\operatorname{cosec} A = \sec COP = \frac{OT'}{r} = \frac{OP}{OM} = \frac{OP}{PN}.$$

Another function is sometimes introduced: if we join $AP$, then $\frac{AP}{r}$ is called the *chord* of $A$ or crd $A$.

11. From the definitions given of the trigonometrical functions a number of connecting relations may be easily deduced; the following are some of the most important, and the student is advised to examine their truth and make himself familiar with them:

$$\sin^2 A + \cos^2 A = 1^*, \ \tan A = \frac{\sin A}{\cos A},$$

$$\sec A = \frac{1}{\cos A}, \ \cot A = \frac{\cos A}{\sin A}, \quad \operatorname{cosec} A = \frac{1}{\sin A}.$$

12. Furthermore, if one of the trigonometrical functions be given, it is not difficult to express all the rest in terms of it. For example, let it be required to express all the functions in terms of the *sine*.

We have $\cos A = \pm\sqrt{1 - \sin^2 A}$,

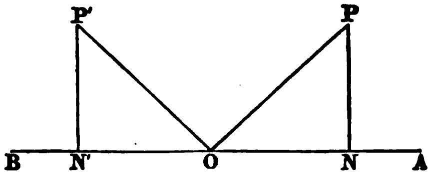
$$\tan A = \pm\frac{\sin A}{\sqrt{1-\sin^2 A}}, \ \sec A = \pm\frac{1}{\sqrt{1-\sin^2 A}},$$

$$\cot A = \pm\frac{\sqrt{1-\sin^2 A}}{\sin A}, \ \operatorname{cosec} A = \frac{1}{\sin A}.$$

* The notation $\sin^2 A$, $\cos^2 A$, &c. represents $(\sin A)^2$, $(\cos A)^2$, &c. or the *square* of $\sin A$, $\cos A$, &c. It appears on the whole to be the most convenient notation, though not universally adopted.

13. Let us here inquire by the way what is the meaning of the ambiguity of the *sign* in the preceding example.

The reason will appear thus: let $AOP$ be the angle of which the sine is given; take $BOP' = AOP$, then it is evident that $\sin AOP' = \sin AOP$, and therefore the given sine may as well belong to $AOP'$ as to $AOP$. But it is not true that $\cos AOP' = \cos AOP$; for $ON$, $ON'$ are drawn on opposite sides of $O$, and therefore if one is positive the other is negative: hence we have $\cos AOP' = -\cos AOP$; and therefore in the above set of formulæ, we must take the upper or lower sign, according as we suppose the given sine to belong to $AOP$ or $AOP'$.




14. The difference between a given angle and two right angles is called its *supplement*. The preceding remarks shew us that the sine of an angle is the sine of its supplement, and that the cosine of an angle is the cosine of its supplement with the sign changed.

15. *To trace the sign of* sin A, *cos* A, *tan* A, *sec* A, *as* A *increases from* $0°$ *to* $360°$. (See the figures of Art. 9.)

By our definition, $\sin A = \dfrac{PN}{OP}$, and has therefore the same sign as $PN$, since there is no reason why $OP$ should ever change sign.

Hence $\sin A$ is positive when $A$ is between $0°$ and $180°$, negative when between $180°$ and $360°$.

$\text{Cos}\, A = \dfrac{ON}{OP}$, and has therefore the same sign as $ON$.

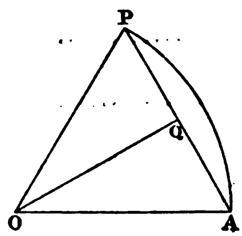
Hence $\cos A$ is positive when $A$ is between $0°$ and $90°$, negative when between $90°$ and $270°$, and positive when between $270°$ and $360°$.

$\text{Tan}\, A = \dfrac{PN}{ON}$, and is therefore positive when $PN$ and $ON$ have the same sign, negative when they have contrary signs.

Hence $\tan A$ is positive when $A$ is between $0^0$ and $90^0$, or between $180^0$ and $270^0$; negative when it is between $90^0$ and $180^0$, or between $270^0$ and $360^0$.

Sec $A = \frac{OP}{ON}$, and has therefore the same sign as $\cos A$.

In like manner may be determined the signs of cot $A$ and cosec $A$. Vers $A$ is always positive.




The sign of chd $A$ may be determined thus. Let $AOP$ be the angle $A$, $AP$ the subtending arc; join $AP$, then chd $A = \frac{AP}{AO}$. Now draw $OQ$ perpendicular to $AP$, and therefore bisecting it; then $OQ$ also bisects the angle $AOP$, and therefore each of the angles $AOQ$, $POQ = \frac{A}{2}$;

$$\therefore \text{chd}\, A = \frac{AP}{AO} = 2\frac{AQ}{AO} = 2\sin\frac{A}{2}.$$

Hence chd $A$ has the same sign as $\sin\frac{A}{2}$: but $\sin\frac{A}{2}$ is positive while $\frac{A}{2}$ is between $0^0$ and $180^0$, or $A$ between $0^0$ and $360^0$, and negative while $\frac{A}{2}$ is between $180^0$ and $360^0$, or $A$ between $360^0$ and $720^0$. Therefore chd $A$ is positive while the revolving line makes its first revolution, negative while it makes its second, and so on*.

*If we adopt the mode of defining the trigonometrical functions given in Art. 10, and consider the radius of the circle to be always a positive quantity as it must be, the change of sign of each of the functions will depend upon that of a single line, namely, the sine upon $PN$, the tangent upon $AT$, and so on; and it is very easy from this mode of definition to determine the change of sign of those functions for which the single line in question is drawn perpendicular to some fixed line, for we have only to assume the functions to be positive in the first quadrant, and then consider them to be positive or negative in the rest, according as they are drawn in the same direction as in the first or in the opposite. To this class belong all the functions except the secant, cosecant, and chord; and it will perhaps be useful to shew how the general principle of using the negative sign as indicative of direction applies in these less simple cases.

1. Speaking of the radius of the circle as unity, we may say that the secant is the line drawn from the centre through the extremity of the arc to meet the line touching the circle at the beginning of the first quadrant; now if this definition be strictly followed in

16. *To determine the change of magnitude of sin A, cos A, tan A, sec A, cot A, cosec A, while A increases from 0° to 360°.*

Retaining the same figures as in the last article, it will be seen, that as

$A$ increases from 0° to 90°, $ON$ decreases from $OP$ to 0,
$PN$ increases from 0 to $OP$,
$A$.............. 90° to 180°, $ON$ increases from 0 to $OP$,
$PN$ decreases from $OP$ to 0,
$A$.............. 180° to 270°, $ON$ decreases from $OP$ to 0,
$PN$ increases from 0 to $OP$,
$A$.............. 270° to 360° $ON$ increases from 0 to $OP$,
$PN$ decreases from $OP$ to 0.

Hence observing the changes of *sign*, as already explained, it is easy to deduce the following table for the changes of sign and value of the functions.

| $A$ between | 0° ...90° | 90° ... 180° | 180° ... 270° | 270° ... 360° |
|---|---|---|---|---|
| sin $A$ | 0 ... 1 | 1 ... 0 | 0 ... − 1 | − 1 ... 0 |
| cos $A$ | 1 ... 0 | 0 ... − 1 | − 1 ... 0 | 0 ... 1 |
| tan $A$ | 0 ... + ∞ | − ∞ ... 0 | 0 ... + ∞ | − ∞ ... 0 |
| sec $A$ | 1 ... + ∞ | − ∞ ... − 1 | − 1 ... − ∞ | + ∞ ... 1 |
| cot $A$ | + ∞ ... 0 | 0 ... − ∞ | + ∞ ... 0 | 0 ... − ∞ |
| cosec $A$ | + ∞ ... 1 | 1 ... + ∞ | − ∞ ... − 1 | − 1 ... − ∞ |

the second quadrant, that is, if in the figure of Art. 10 we join $OP'$ and produce it, we shall never make it meet the line touching the circle at $A$; therefore we must suppose it drawn in the reverse direction and must account it negative.

2. Similar remarks apply to the cosecant.

3. The chord is the portion of the chord drawn through $A$ (fig. Art. 10) and the extremity of the arc, intercepted by the circle; now if we suppose an indefinite line to revolve round one extremity $A$, it is evident that a certain portion of it will be intercepted by the circle while the line revolves through two right angles, that is, while $OP$ revolves through four right angles, but after that if the line continue to revolve no portion will be intercepted; consequently there will be no chord from 360° to 720° unless we suppose the above revolving line to be produced backwards, and the chords so formed will be rightly accounted negative.

The preceding conclusions are in accordance with those in the text.

Another mode of considering this question is to regard the revolving line as indefinite on both sides of the point about which it revolves, and then the positive and negative intercepted lines may be discriminated by observing on which side of the fixed point of revolution the intercepted portion lies; if it lie on the same side as that which was intercepted in the case of the first quadrant it will be positive, if otherwise negative.

The student is recommended to make himself as familiar as possible with the results of the preceding table: and it may be observed, to aid him in doing so, that if he becomes well acquainted with the changes of the *sine* and *cosine*, those of the other functions are at once deducible.

It will be remarked that the trigonometrical functions change sign in passing through 0 and $\infty$, and for no other values*.

17. The values of vers $A$ and chd $A$ have not yet been given. They are of no great importance, as those functions may always be replaced by $1 - \cos A$, and $2 \sin \frac{A}{2}$ respectively; they are however as follows:

| $A$ between | $0^\circ \ldots 90^\circ$ | $90^\circ \ldots 180^\circ$ | $180^\circ \ldots 270^\circ$ | $270^\circ \ldots 360^\circ$ |
|---|---|---|---|---|
| vers $A$ | $0 \ldots 1$ | $1 \ldots 2$ | $2 \ldots 1$ | $1 \ldots 0$ |

| $A$ between | $0^\circ \ldots 180^\circ$ | $180^\circ \ldots 360^\circ$ | $360^\circ \ldots 540^\circ$ | $540^\circ \ldots 720^\circ$ |
|---|---|---|---|---|
| chord $A$ | $0 \ldots 2$ | $2 \ldots 0$ | $0 \ldots -2$ | $-2 \ldots 0$ |

We have already seen that $\sin A = \sin (180^\circ - A)$ and that $\cos A = -\cos (180^\circ - A)$ (Art. 14); we shall now prove more general propositions of the same kind.

18. *To prove that,* n *being any integer,*

$$\sin A = \pm \sin (4n \,.\, 90^\circ \pm A) = \pm \sin \{(4n + 2)\, 90^\circ \mp A\}.$$

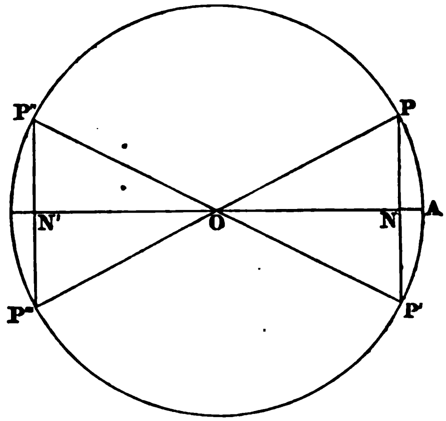
It is manifest that none of the trigonometrical functions (except the chord) are altered by supposing the angle to be increased by any number of complete revolutions of the revolving line; that is to say, $\sin A = \sin (4n \,.\, 90^\circ + A.)$

* This is sometimes a very convenient principle upon which to determine the values for which a trigonometrical quantity changes its sign. For example, let it be required to determine the values of $A$ for which $\sin A + \cos A$ changes from + to −, or from − to +. A little consideration will shew that for $A = 135^\circ$, $\sin A + \cos A = 0$; and that the same is true for $A = 315^\circ$. Consequently from $0^\circ$ to $135^\circ$ the quantity is positive, from $135^\circ$ to $315^\circ$ it is negative, and again from $315^\circ$ to $360^\circ$ positive. Let the student discuss in the same manner $\tan A - \cot A$.

Again, if we take $P'OA = -POA$, it is evident that $P'N = PN$, and

$$\therefore \sin A = -\sin(-A)$$
$$= -\sin(4n \cdot 90^\circ - A)$$

by what precedes.




Again, if we produce $PO, P'O$, to $P'''$, $P''$ it may be seen that

$$\sin(180^\circ - A) = \sin A,$$
$$\text{and } \sin(180^\circ + A) = -\sin A.$$

Hence
$$\sin A = \pm \sin(2 \cdot 90^\circ \mp A)$$
$$= \pm \sin\{(4n + 2)\,90^\circ \mp A\}.$$

Other propositions may be demonstrated in like manner: as for example,

$$\cos A = \cos(720^\circ \pm A),$$
$$\cos A = -\cos(540^\circ \pm A),$$
$$\tan A = \pm \tan(180^\circ \pm A).$$

## ON FORMULÆ INVOLVING MORE THAN ONE ANGLE.

19. *Given the sines and cosines of two angles, to find the sine and cosine of their sum or difference.*

Let $POQ$, $QOM$ be any two angles, which call $A$ and $B$ respectively.

From any point $P$ in $OP$ draw $PQ$ perpendicular to $OQ$, and from $P$, $Q$ draw $PN$, $QM$ perpendicular to $OM$, and $QR$ perpendicular to $PN$.

It will be seen that

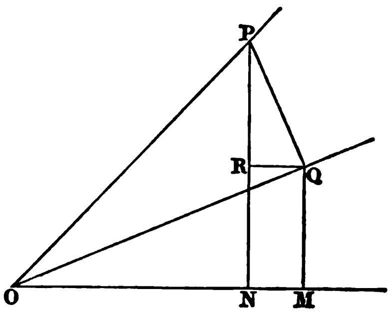
$QPR = 90^0 - PQR$

$= RQO = QOM = B$;

then $\sin(A+B) = \sin PON$

$= \frac{PN}{OP} = \frac{PR + QM}{OP}$

$= \frac{PQ}{OP} \cdot \frac{PR}{PQ} + \frac{OQ}{OP} \cdot \frac{QM}{OQ}$




$= \sin A \cos B + \cos A \sin B \ldots\ldots(1).$

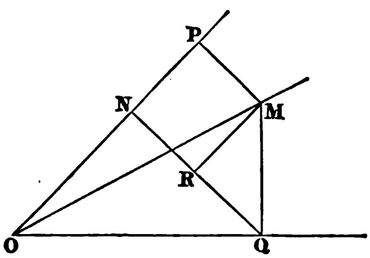
Again, $\cos(A+B) = \cos PON = \frac{ON}{OP} = \frac{OM - QR}{OP}$

$= \frac{OQ}{OP} \cdot \frac{OM}{OQ} - \frac{PQ}{OP} \cdot \frac{QR}{PQ}$

$= \cos A \cos B - \sin A \sin B \ldots\ldots(2).$

For the sine and cosine of the *difference* of two angles we must make a new construction.

Let $POQ = A$, $QOM = B$ as before, then $POM = A - B$. From any point $M$ in $OM$ draw the lines $MP$, $MQ$ perpendicular to $OP$ and $OQ$ respectively, $QN$ perpendicular to $OP$, and $MR$ to $QN$. It is evident that $MQR = A$.




Then $\sin(A-B) = \sin POM = \frac{PM}{OM} = \frac{QN - QR}{OM}$

$= \frac{QN}{OQ} \cdot \frac{OQ}{OM} - \frac{QR}{MQ} \cdot \frac{MQ}{OM}$

$= \sin A \cos B - \cos A \sin B \ldots\ldots(3).$

Again, $\cos(A-B) = \cos POM = \frac{OP}{OM} = \frac{ON + MR}{OM}$

$= \frac{ON}{OQ} \cdot \frac{OQ}{OM} + \frac{MR}{MQ} \cdot \frac{MQ}{OM}$

$= \cos A \cos B + \sin A \sin B \ldots\ldots(4).$

20. It will be observed, that the preceding formulæ have been proved by means of figures which suppose both $A$ and $B$, as well as $A + B$ and $A - B$, to be each less than a right angle; nevertheless we are justified in concluding that the same formulæ will hold in all cases, provided the proper signs be given to the quantities which enter: and herein consists one great advantage of the mode of denoting the directions of lines by their signs, that when any formula has been established for a standard case in which all the lines are positive, the same may be safely assumed to be true in all other cases*.

As an example of what has been here remarked, it may be observed, that the formulæ (3) and (4) just proved may be deduced from the formulæ (1) and (2) by changing the sign of the angle $B$. For we have from (1),

$$\sin(A + B) = \sin A \cos B + \cos A \sin B;$$

now write $-B$ for $B$, and we have

$$\begin{aligned}\sin(A - B) &= \sin A \cos(-B) + \cos A \sin(-B)\\ &= \sin A \cos B - \cos A \sin B,\end{aligned}$$

which agrees with (3); and similarly (4) may be deduced from (2).

But still further, (2) may be deduced from (1): for we have

$$\begin{aligned}\cos(A + B) &= \sin\{(90^\circ - A) - B\}\\ &= \sin(90^\circ - A)\cos(-B) + \cos(90^\circ - A)\sin(-B)\\ &= \cos A \cos B - \sin A \sin B.\end{aligned}$$

Hence it appears, that the only formula which it is quite necessary to establish by reference to a geometrical figure, is the fundamental formula (1).

21. By making $B = A$ we obtain the following formulæ:

$$\sin 2A = 2\sin A \cos A,$$

$$\cos 2A = \cos^2 A - \sin^2 A.$$

* The student will find it a useful exercise to demonstrate any of the formulæ (1), (2), (3), (4) in particular cases in which the preceding conditions are not fulfilled; as for instance, when $A$ is $> 90^\circ$, $B < 90^\circ$, and $A + B < 180^\circ$. In doing so let him draw a figure adapted to the particular case.

which last formula, in consequence of the relation

$$\cos^2 A + \sin^2 A = 1,$$

may be put under either of the following forms,

$$\cos 2A = 2\cos^2 A - 1,$$

$$\cos 2A = 1 - 2\sin^2 A.$$

Also, conversely, we have the following useful formulæ,

$$\cos A = \pm\sqrt{\frac{1+\cos 2A}{2}},$$

$$\sin A = \pm\sqrt{\frac{1-\cos 2A}{2}}.$$

We may observe that the sign $\pm$ necessarily attaches to these two formulæ; and when the limits between which $A$ lies are known, the proper sign may be given. Thus, if $A$ lie between $0^0$ and $90^0$,

$$\cos A = +\sqrt{\frac{1+\cos 2A}{2}},\ \sin A = +\sqrt{\frac{1-\cos 2A}{2}},$$

if $A$ lie between $90^0$ and $180^0$,

$$\cos A = -\sqrt{\frac{1+\cos 2A}{2}},\ \sin A = +\sqrt{\frac{1-\cos 2A}{2}},$$

if $A$ lie between $180^0$ and $270^0$,

$$\cos A = \sqrt{\frac{1+\cos 2A}{2}},\ \sin A = -\sqrt{\frac{1-\cos 2A}{2}},$$

and if $A$ lie between $270^0$ and $360^0$,

$$\cos A = \sqrt{\frac{1+\cos 2A}{2}},\ \sin A = -\sqrt{\frac{1-\cos 2A}{2}}.$$

We have expressed $\sin A$ and $\cos A$ in terms of $\cos 2A$; we may also obtain expressions for them in terms of $\sin 2A$, as follows:

$$1 + \sin 2A = \cos^2 A + \sin^2 A + 2\sin A\cos A = (\cos A + \sin A)^2,$$

$$1 - \sin 2A = \cos^2 A + \sin^2 A - 2\sin A\cos A = (\cos A - \sin A)^2;$$

$$\left.\begin{aligned}\therefore\ \cos A + \sin A &= \pm\sqrt{1 + \sin 2A},\\ \cos A - \sin A &= \pm\sqrt{1 - \sin 2A},\end{aligned}\right\}\ldots(a).$$

and
$$\cos A = \pm\tfrac{1}{2}\sqrt{1 + \sin 2A} \pm \tfrac{1}{2}\sqrt{1 - \sin 2A},$$
$$\sin A = \pm\tfrac{1}{2}\sqrt{1 + \sin 2A} \mp \tfrac{1}{2}\sqrt{1 - \sin 2A},$$
the expressions required.

There is greater difficulty in determining the proper signs to be used in this case than in the preceding; but the following considerations will lead us to the point which we desire*.

From $0^\circ$ to $45^\circ$, $\sin A$ is positive, $\cos A$ is positive, and $\cos A$ is $> \sin A$; therefore, looking to the equations $(a)$, we see that the upper sign of each radical must be taken.

From $45^\circ$ to $135^\circ$, for like reasons, $\cos A + \sin A$ is positive, and $\cos A - \sin A$ negative.

From $135^\circ$ to $225^\circ$, $\cos A + \sin A$ is negative, and $\cos A - \sin A$ negative.

From $225^\circ$ to $315^\circ$, $\cos A + \sin A$ is negative, and $\cos A - \sin A$ positive.

From $315^\circ$ to $360^\circ$, $\cos A + \sin A$ is positive, and $\cos A - \sin A$ positive.

Hence we have the following table of signs:

| | $0^\circ$ to $45^\circ$ | | $45^\circ$ to $135^\circ$ | | $135^\circ$ to $225^\circ$ | | $225^\circ$ to $315^\circ$ | | $315^\circ$ to $360^\circ$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\cos A$ | + | + | + | − | − | − | − | + | + | + |
| $\sin A$ | + | − | + | + | − | + | − | − | + | − |

22. *To express* $\tan (A \pm B)$ *in terms of* $\tan A$ *and* $\tan B$.

We have

$$\tan (A \pm B) = \frac{\sin (A \pm B)}{\cos (A \pm B)}$$
$$= \frac{\sin A \cos B \pm \cos A \sin B}{\cos A \cos B \mp \sin A \sin B}.$$

* The signs of these radicals may also be determined upon the principle used in the note upon page 119, that is, the principle that the radicals will change their signs when the quantities beneath them become zero.

(dividing numerator and denominator by $\cos A \cos B$)

$$= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}.$$

Making $B = A$, we obtain the formula

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}.$$

Hence also

$$\cot (A \pm B) = \frac{1 \mp \tan A \tan B}{\tan A \pm \tan B} = \frac{\cot A \cot B \mp 1}{\cot B \pm \cot A},$$

$$\text{and } \cot 2A = \frac{\cot^2 A - 1}{2 \cot A}.$$

23. The following formulæ are of great service in trigonometrical investigations.

We have

$$A = \frac{A+B}{2} + \frac{A-B}{2}, \quad B = \frac{A+B}{2} - \frac{A-B}{2};$$

$$\therefore \sin A = \sin \frac{A+B}{2} \cos \frac{A-B}{2} + \cos \frac{A+B}{2} \sin \frac{A-B}{2},$$

$$\sin B = \sin \frac{A+B}{2} \cos \frac{A-B}{2} - \cos \frac{A+B}{2} \sin \frac{A-B}{2},$$

$$\cos A = \cos \frac{A+B}{2} \cos \frac{A-B}{2} - \sin \frac{A+B}{2} \sin \frac{A-B}{2},$$

$$\cos B = \cos \frac{A+B}{2} \cos \frac{A-B}{2} + \sin \frac{A+B}{2} \sin \frac{A-B}{2}.$$

Hence

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} \quad \text{......(1)}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2} \quad \text{......(2)}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} \quad \text{......(3)}$$

$$\cos B - \cos A = 2 \sin \frac{A+B}{2} \sin \frac{A-B}{2} \quad \text{......(4)}$$

24. The formulæ which have been investigated in the last few articles may be easily extended to functions of *three* or more angles. For instance,

$$\sin(A+B+C) = \sin(A+B)\cos C + \cos(A+B)\sin C$$
$$= (\sin A\cos B + \cos A\sin B)\cos C + (\cos A\cos B - \sin A\sin B)\sin C$$
$$= \sin A\cos B\cos C + \sin B\cos A\cos C$$
$$+ \sin C\cos A\cos B - \sin A\sin B\sin C.$$

Similarly we may express $\cos(A+B+C)$, $\tan(A+B+C)$, &c., in terms of the trigonometrical functions of the simple angles.

In like manner we may express $\sin 3A$, $\cos 3A$.

$$\sin 3A = \sin(A+2A) = \sin A\cos 2A + \cos A\sin 2A$$
$$= \sin A(1-2\sin^2 A) + 2\sin A\cos^2 A$$
$$= \sin A\{1-2\sin^2 A + 2 - 2\sin^2 A\}$$
$$= 3\sin A - 4\sin^3 A;$$

$$\cos 3A = \cos(A+2A) = \cos A\cos 2A - \sin A\sin 2A$$
$$= \cos A(2\cos^2 A - 1) - 2\sin^2 A\cos A$$
$$= \cos A\{2\cos^2 A - 1 - 2 + 2\cos^2 A\}$$
$$= 4\cos^3 A - 3\cos A.$$

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It will be convenient for purposes of reference to collect in one table the formulæ which have been investigated.

$$\sin(A+B) = \sin A\cos B + \cos A\sin B, \ldots\ldots (1)$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B, \ldots\ldots (2)$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B, \ldots\ldots (3)$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B, \ldots\ldots (4)$$
$$\sin 2A = 2\sin A\cos A, \text{ or } \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} \ldots\ldots (5)$$
$$\left.\begin{array}{l}\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A,\\ \text{or } \cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2} = 2\cos^2\frac{A}{2} - 1 = 1 - 2\sin^2\frac{A}{2}\end{array}\right\} \quad (6)$$

$$\cos A = \pm \sqrt{\frac{1 + \cos 2A}{2}}, \text{ ......................} (7)$$

$$\sin A = \pm \sqrt{\frac{1 - \cos 2A}{2}}, \text{ ......................} (8)$$

$$\cos A = \pm \tfrac{1}{2} \sqrt{1 + \sin 2A} \pm \tfrac{1}{2} \sqrt{1 - \sin 2A}, \text{......} (9)$$

$$\sin A = \pm \tfrac{1}{2} \sqrt{1 + \sin 2A} \mp \tfrac{1}{2} \sqrt{1 - \sin 2A}, \text{......} (10)$$

$$\tan (A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}, \text{ ......................} (11)$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}, \text{ or } \tan A = \frac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}}, \text{ ...} (12)$$

$$\cot (A \pm B) = \frac{\cot A \cot B \mp 1}{\cot B \pm \cot A}, \text{ ......................} (13)$$

$$\cot 2A = \frac{\cot^2 A - 1}{2 \cot A}, \text{ or } \cot A = \frac{\cot^2 \frac{A}{2} - 1}{2 \cot \frac{A}{2}}, \text{ ......} (14)$$

$$\sin A + \sin B = 2 \sin \frac{A + B}{2} \cos \frac{A - B}{2}, \text{ ................} (15)$$

$$\sin A - \sin B = 2 \cos \frac{A + B}{2} \sin \frac{A - B}{2}, \text{ ................} (16)$$

$$\cos A + \cos B = 2 \cos \frac{A + B}{2} \cos \frac{A - B}{4}, \text{ ................} (17)$$

$$\cos B - \cos A = 2 \sin \frac{A + B}{2} \sin \frac{A - B}{2}, \text{ ................} (18)$$

$$\sin 3A = 3 \sin A - 4 \sin^3 A, \text{ ......................} (19)$$

$$\cos 3A = 4 \cos^3 A - 3 \cos A. \text{ ......................} (20)$$

## DETERMINATION OF THE NUMERICAL VALUES OF THE TRIGONOMETRICAL FUNCTIONS OF ANGLES.

25. The value of the trigonometrical functions of certain angles may be expressed with great facility and simplicity, that of others must be found by a laborious course of calculation. A few which present the most simple investigation are here given.

*To find the value of sin 45°.*

In general $\sin^2 A + \cos^2 A = 1.$

$$\text{Let } A = 45^\circ;\ \therefore\ \cos A = \cos 45^\circ = \sin 45^\circ;$$

$$\therefore\ 2\sin^2 45^\circ = 1,$$

$$\sin^2 45^\circ = \tfrac{1}{2},$$

$$\sin 45^\circ = \frac{1}{\sqrt{2}};$$

the positive sign of the radical is taken, because we know that sin 45° is positive.

Hence also $\cos 45^\circ = \dfrac{1}{\sqrt{2}}$, and $\tan 45^\circ = \cot 45^\circ = 1.$

*To find the value of sin 30°.*

$$\cos 30^\circ = \sin 60^\circ = 2\sin 30^\circ \cos 30^\circ;$$

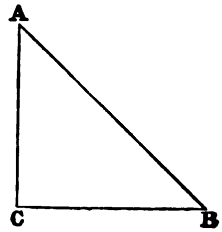
$$\therefore\ 2\sin 30^\circ = 1,$$

$$\sin 30^\circ = \tfrac{1}{2}.$$

Also $\cos 30^\circ = \sqrt{1 - \sin^2 30^\circ} = \dfrac{\sqrt{3}}{2}$, and $\tan 30^\circ = \dfrac{1}{\sqrt{3}}$.

The values of sin 45° and sin 30° may be obtained very simply by a geometrical figure, thus:

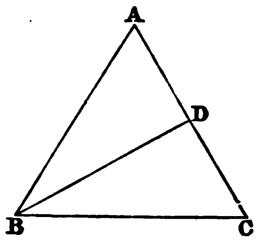
Let $ACB$ be an isosceles triangle and having a right angle at $C$. Then each of the angles $A$ and $B$ is an angle of $45^0$.



$$\therefore \sin 45^0 = \frac{BC}{AB} = \frac{BC}{\sqrt{AC^2 + BC^2}}$$

$$= \frac{BC}{\sqrt{2BC^2}} = \frac{1}{\sqrt{2}}.$$

Again, let $ABC$ be an equilateral triangle, and draw $BD$ perpendicular to, and therefore bisecting $AC$. Then since the angles $A$, $B$, $C$, are equal to each other and together equal to $180^0$, each of them is $60^0$, and therefore $ABD$, which manifestly is half of $ABC$, $= 30^0$.



$$\therefore \sin 30^0 = \frac{AD}{AB} = \frac{1}{2}.$$

*To find the value of* $\sin 18^0$.

$$36^0 = 90^0 - 54^0;$$

$$\therefore \sin 36^0 = \cos 54^0.$$

We have seen (Art. 24) that $\cos 3A = 4\cos^3 A - 3\cos A$; also, $36^0 = 2 \times 18^0$, and $54^0 = 3 \times 18^0$;

$$\therefore 2 \sin 18^0 \cos 18^0 = 4 \cos^3 18^0 - 3 \cos 18^0,$$

$$2 \sin 18^0 = 4 \cos^2 18^0 - 3 = 1 - 4 \sin^2 18^0,$$

$$\sin^2 18^0 + \frac{\sin 18^0}{2} = \frac{1}{4}.$$

Completing the square,

$$\sin^2 18^0 + \frac{\sin 18^0}{2} + \frac{1}{16} = \frac{1}{4} + \frac{1}{16} = \frac{5}{16},$$

$$\sin 18^0 = \frac{-1 + \sqrt{5}}{4};$$

the positive sign is given to the radical, because we know that $\sin 18^0$ is positive.

This result may also be deduced from the figure used in Euclid, IV. 10; for let $BAD$ be the triangle constructed in that proposition, and $C$ the point in $AB$ such that $AB.BC = AC^2$; then by the nature of the triangle $BAD$,

$$\text{angle } BAD = \frac{180^0}{5} = 36^0;$$

$$\therefore 2 \sin 18^0 = \text{chd } 36^0 = \text{chd } BAD = \frac{BD}{AB} = \frac{AC}{AB}.$$

Let $AC = x$, $AB = r$, then $\sin 18^0 = \dfrac{x}{2r}$,

$$\text{and } r(r - x) = x^2,$$

$$\therefore x^2 + rx + \frac{r^2}{4} = \frac{5r^2}{4}$$

$$\text{or} \qquad x = \frac{-1 + \sqrt{5}}{2} r;$$

$$\therefore \sin 18^0 = \frac{-1 + \sqrt{5}}{4}, \text{ as before.}$$

By means of the preceding values, we may investigate a variety of others. For instance, if it were required to find $\sin 12^0$, we should have

$$\begin{aligned} \sin 12^0 &= \sin (30^0 - 18^0), \\ &= \sin 30^0 \cos 18^0 - \cos 30^0 \sin 18^0; \end{aligned}$$

and consequently $\sin 12^0$ becomes known. In short, we can find the values of the trigonometrical functions of any angles, which are combinations of those which have been discussed.

26. The values of the trigonometrical functions of all angles from $1'$ up to $90^0$ may be calculated by methods not here explained, and arranged in tables for the purposes of practical application. Still more useful are tables containing the *logarithms* of these values; the student who is desirous of understanding the mode of constructing such tables is referred to the treatise of *Snowball* or *Hymers*, or any other complete treatise on Trigonometry.

## ON THE SOLUTION OF TRIANGLES.

27. A triangle consists of six parts, viz., three sides and three angles: three of these being given, the others may be found, unless the three *angles* be the three given parts, in which case nothing further can be found. The angles of a triangle are not in fact *independent parts*, their sum being always two right angles; it may be said, that if three *independent parts* be given, the others may in general be found.

28. Before we engage ourselves with the solution of a triangle, we must investigate certain formulæ connecting the parts of a triangle.

We shall denote the angles of a triangle by $A$, $B$, $C$, the sides respectively opposite to them by $a$, $b$, $c$.

29. *The sides of a triangle are proportional to the sines of the opposite angles.*

Fig. I.

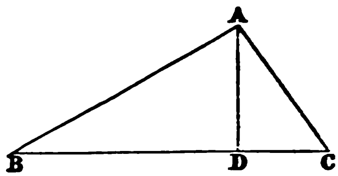
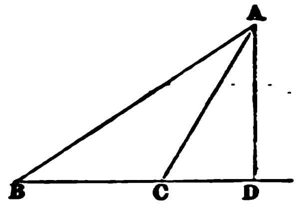



Fig. II.




Let $ABC$ be a triangle. From any angular point $A$ let fall the perpendicular $AD$ on the opposite side (fig. 1), or that side produced (fig. 2).

Then in the first case, $AD = c \sin B = b \sin C$;
in the second, $AD = c \sin B = b \sin (180^\circ - C) = b \sin C$;

$$\text{therefore in both, } \frac{\sin B}{b} = \frac{\sin C}{c}.$$

In like manner,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c};$$

which proves the theorem.

30. *To express the cosine of an angle of a triangle in terms of the sides.*

With the same figures as in the last proposition, we have in fig. 1,

$$a = BD + DC = c \cos B + b \cos C,$$

in fig. 2,

$$a = BD - DC = c \cos B - b \cos (180^\circ - C) = c \cos B + b \cos C;$$

therefore in both cases,

$$\frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2}{abc},$$

in like manner, $\dfrac{\cos C}{c} + \dfrac{\cos A}{a} = \dfrac{b^2}{abc};$

and $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} = \dfrac{c^2}{abc};$

adding the last two of these equations and subtracting the first, we have

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

which is the expression required*.

* It may be remarked that the formula of this article is deducible from those of the preceding, and conversely. Thus assuming that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

and that $A + B + C = 180^\circ$,

we have $\sin A = \sin (B + C) = \sin C \cos B + \sin B \cos C;$

$$\therefore a = c \cos B + b \cos C,$$

which is the fundamental equation in the text; the remainder of the process is of course the same.

To deduce the formulæ of Art. 29 from that of Art. 30, it will be seen that in Art. 32 the following formula is proved,

$$\sin A = \frac{2}{bc}\sqrt{S(S-a)(S-b)(S-c)},$$

where $S = \dfrac{a+b+c}{2};$

$$\therefore \frac{\sin A}{a} = \frac{2}{abc}\sqrt{S(S-a)(S-b)(S-c)};$$

This formula may be written thus:

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

which equation, as it is easy to see, includes in itself Props. XII. and XIII. of Euclid, Book II.

31. *To express* $\sin \frac{A}{2}$, $\cos \frac{A}{2}$, $\tan \frac{A}{2}$ *in terms of the sides.*

By Art. 21, $\sin^2 \frac{A}{2} = \frac{1 - \cos A}{2} = \frac{2bc - b^2 - c^2 + a^2}{4bc}$,

$$= \frac{a^2 - (b - c)^2}{4bc} = \frac{(a - b + c)(a + b - c)}{4bc}.$$

Let $a + b + c = 2S$,

$$\therefore\ a - b + c = 2(S - b),$$

$$a + b - c = 2(S - c),$$

and $\sin \frac{A}{2} = \sqrt{\frac{(S - b)(S - c)}{bc}}$.

In like manner,

$$\cos^2 \frac{A}{2} = \frac{1 + \cos A}{2} = \frac{2bc + b^2 + c^2 - a^2}{4bc},$$

$$= \frac{(b + c)^2 - a^2}{4bc} = \frac{(a + b + c)(b + c - a)}{4bc},$$

$$= \frac{S(S - a)}{bc};$$

$$\therefore\ \cos \frac{A}{2} = \sqrt{\frac{S(S - a)}{bc}}.$$

Hence also we have

$$\tan \frac{A}{2} = \sqrt{\frac{(S - b)(S - c)}{S(S - a)}}.$$

Now this expression for $\frac{\sin A}{a}$ involves $a$, $b$, $c$ symmetrically, and therefore we should arrive at the same result if we had expressed $\frac{\sin B}{b}$ or $\frac{\sin C}{c}$ in terms of $a$, $b$, and $c$. Hence it follows that $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

32. *To express sin A in terms of the sides.*

This is done at once by means of the expressions just proved: for

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2},$$

$$= \frac{2}{bc} \sqrt{S(S-a)(S-b)(S-c)}.$$

This expression has the advantage of being adapted to *logarithmic computation*, that is to say, it consists of *factors*. The formula for the cosine (Art. 30) has not this advantage.

33. *To prove the following formula,*

$$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}.$$

We have, by Art. 29,

$$\frac{a}{b} = \frac{\sin A}{\sin B};$$

$$\therefore \frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B}$$

$$= \frac{2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}}{2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}}, \text{ by Art. 23,}$$

$$= \frac{\tan \frac{A-B}{2}}{\tan \frac{A+B}{2}};$$

but $A + B = 180^\circ - C$,

$$\therefore \frac{A+B}{2} = 90^\circ - \frac{C}{2},$$

and $\tan \frac{A+B}{2} = \cot \frac{C}{2}$.

Substituting in the equation before obtained, we have

$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}.$$

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Having established the preceding necessary formulæ, we ow proceed to the solution of triangles.

34. *Let two angles and the side between them* (A, C, b) *be iven.*

The other angle is known at once, because

$$B = 180^\circ - A - C.$$

Again, we have

$$a = b\frac{\sin A}{\sin B},$$

$$\text{and } c = b\frac{\sin C}{\sin B};$$

which determine $a$ and $c$.

35. *Let two angles and a side opposite to one of them* (A, C, a) *be given.*

As before, the third angle is known, because

$$B = 180^\circ - A - C.$$

Also,

$$b = a\frac{\sin B}{\sin A},$$

$$c = a\frac{\sin C}{\sin A},$$

which determine $b$ and $c$.

36. *Let two sides and the included angle* (C, a, b) *be given.*

We determine the other angles thus,

$$A + B = 180^\circ - C.$$

Again, by Art. 33,

$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2},$$

which determines $A-B$: thus $A+B$ and $A-B$ are both known; and therefore $A$ and $B$, which are $\frac{A+B}{2}+\frac{A-B}{2}$ and $\frac{A+B}{2}-\frac{A-B}{2}$ respectively, are also known.

To determine $c$ we have

$$c = a\frac{\sin C}{\sin A}.$$

37. There is another mode of solving the triangle in this case.

Since $$\cos C = \frac{a^2+b^2-c^2}{2ab},$$

we have $c^2 = a^2+b^2-2ab\cos C$;

and this equation in fact determines $c$, but in its present state it would be practically of no use because it is *not adapted to logarithmic computation;* we can however modify it in such a manner as to render it suitable, as follows:

$$\begin{aligned}
c^2 &= a^2+b^2-2ab\cos C \\
&= (a^2+b^2)\left(\cos^2\frac{C}{2}+\sin^2\frac{C}{2}\right)-2ab\left(\cos^2\frac{C}{2}-\sin^2\frac{C}{2}\right) \\
&= (a-b)^2\cos^2\frac{C}{2}+(a+b)^2\sin^2\frac{C}{2} \\
&= (a-b)^2\cos^2\frac{C}{2}\left\{1+\left(\frac{a+b}{a-b}\right)^2\tan^2\frac{C}{2}\right\}.
\end{aligned}$$

Since the tangent of an angle may be of any magnitude, there will be an angle the tangent of which is $\frac{a+b}{a-b}\tan\frac{C}{2}$; let $\theta$ be such an angle, so that

$$\tan\theta = \frac{a+b}{a-b}\tan\frac{C}{2},$$

or that $\log\tan\theta = \log(a+b)-\log(a-b)+\log\tan\frac{C}{2}$.

The value of $\theta$ is found by looking in the tables and finding an angle which has for its logarithmic tangent the

preceding quantity, and may therefore be now supposed to be *known*.

Also we have

$$c^2 = (a-b)^2 \cos^2 \frac{C}{2} (1 + \tan^2 \theta)$$

$$= (a-b)^2 \cos^2 \frac{C}{2} \sec^2 \theta,$$

$$\text{or } c = (a-b) \frac{\cos \frac{C}{2}}{\cos \theta},$$

which equation determines $c$.

The sides $a, b, c$ being all known, $A$ and $B$ may be determined by any of the expressions given in Arts. (30), (31), (32).

It may be noticed, that $\tan \theta = \cot \frac{A-B}{2}$, or $\theta = 90^\circ - \frac{A-B}{2}$;

hence $c = (a-b) \dfrac{\cos \frac{C}{2}}{\sin \frac{A-B}{2}}$, a result which may easily be demonstrated directly. Hence it appears that the preceding process is equivalent to finding $A - B$, which was also the first step in the other method of solving the triangle.

38. Having spoken of a formula being *adapted to logarithmic computation*, we will explain what is meant by the term. A formula is said to be so adapted when it consists entirely of *factors*, or quantities *multiplied together;* the factors may however consist of more than one simple term, as for instance in the preceding article, $a - b$ is one of the factors of our final expression. When a formula then consists of factors, the calculations by logarithms may be effected by mere addition and subtraction, but if otherwise, logarithms can scarcely be applied at all. Let us consider, for example, how $c$ could be calculated from the formula,

$$c^2 = a^2 + b^2 - 2ab \cos C,$$

without the process which has been given. The three terms

$a^2$, $b^2$, and $2ab \cos C$, would necessarily be calculated separately, and there would be a fourth operation for determining $c$, when the value of $c^2$ was known; thus the process would be extremely tedious. In practice, therefore, no trigonometrical formula can be considered as being of any utility, unless it is adapted to logarithmic computation.

The process of adapting a formula to logarithmic computation, which has been introduced in the preceding article, is one of very frequent use. The angle $\theta$, which has been employed to assist the calculation, is called a *subsidiary angle.* It will be seen that the possibility of facilitating calculations by this means, arises from the fact of our possessing tables containing all the trigonometrical functions of the same angle, so that as soon as one function of an angle is known all the others become known; for instance, in the case we have been considering, as soon as a certain quantity was fixed upon as representing the *tangent* of an angle, the *cosine* of that angle was known by inspection of the table, and the calculation was spared by which it would have been necessary to determine the cosine from the tangent. The adaptation of formulæ to logarithmic computation is a matter not of rule, but of ingenuity, and frequently a formula may be adapted in various ways equally good; we must however be careful to ascertain that the supposition we make involves no absurdity; for instance, we may not assume a quantity to be equal to $\cos\theta$, ($\theta$ being the subsidiary angle,) unless we have ascertained that the quantity is not greater than unity; and so in other instances.

We subjoin a few examples of the adaptation of formulæ to logarithmic computation.

Ex. 1. $x = \sqrt{a + \sqrt{a^2 - b^2}} + \sqrt{a - \sqrt{a^2 - b^2}}$, where $b$ is less than $a$.

Assume $b = a \sin\theta$, which equation determines $\theta$;

$$\therefore\ x = \sqrt{a + a\cos\theta} + \sqrt{a - a\cos\theta}$$

$$= \sqrt{2a}\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)$$

$$= 2\sqrt{a}\left(\frac{1}{\sqrt{2}}\cos\frac{\theta}{2} + \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}\right)$$

$$= 2\sqrt{a}\sin\left(\frac{\theta}{2} + 45^\circ\right),$$

which form is adapted to logarithmic computation.

Ex. 2. $x = \cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\gamma$

$$= \cos\alpha\cos\beta\,(1 + \tan\alpha\tan\beta\cos\gamma).$$

Assume $\tan\alpha\tan\beta\cos\gamma = \tan\theta$;

$$\therefore\ x = \cos\alpha\cos\beta\,(1 + \tan\theta),$$

$$= \cos\alpha\cos\beta\,\frac{\cos\theta + \sin\theta}{\cos\theta},$$

$$= \sqrt{2}\cos\alpha\cos\beta\,\frac{\sin(\theta + 45^\circ)}{\cos\theta},$$

which is in the form required.

39. We may here make a remark respecting the tables of logarithmic trigonometrical functions, which, if we had troubled the student with a complete account of the formation of such tables, would have found a more fitting place elsewhere.

It appeared from the account of logarithms given in the treatise on Algebra, (page 102, Art. 128) that the logarithms of numbers less than unity have *negative characteristics.* Now all sines and cosines are less than unity, (except $\sin 90^0$ and $\cos 90^0$,) and therefore their logarithms have negative characteristics; but it is not convenient to register such quantities in tables, and therefore it is usual to add 10 to each of the logarithmic functions, and thus the characteristic $-1$ is replaced by 9. This is merely matter of convenient arrangement, but it entails the following precaution, that when the logarithms of numbers and those of trigonometrical functions occur in the same equation, 10 *must be subtracted from each logarithmic function of an angle.*

For instance, suppose we had the equation

$$b = a\sin C,$$

in which $a$ and $C$ are given and $b$ is to be found; we should take the logarithms thus,

$$\log b = \log a + \log \sin C - 10,$$

instead of

$$\log b = \log a + \log \sin C.$$

Or we may make the distinction between the two kinds of logarithms more manifest by a difference of notation, and may write the preceding formula thus,

$$\log_{10} b = \log_{10} a + L \sin C - 10,$$

where $\log_{10}$ indicates the logarithm of a number to base 10, and $L$ the logarithm of a trigonometrical function to the same base when 10 has been added to it.

40. *Let two sides and an angle opposite to one of them* (a, b, A) *be given.*

To determine $B$, we have

$$\sin B = \frac{b}{a} \sin A;$$

$C$ is then known from the formula

$$C = 180^\circ - A - B,$$

and $c$ from

$$c = a \frac{\sin C}{\sin A}.$$

41. The solution of the triangle in this case however is not without ambiguity; for the equation

$$\sin B = \frac{b}{a} \sin A,$$

does not determine *one* angle but *two*, because

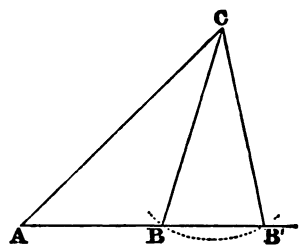
$$\sin B = \sin (180^\circ - B),$$

and the question is whether there is any test to guide us in choosing one of the values rather than the other.

Now $180^\circ - B = A + C$, and therefore $B$ and $180^\circ - B$ cannot both be less than $A$; but the greater side is opposite the greater angle, (Euclid, I. 18) consequently, if $b$ be *less* than $a$, $B$ must be less than $A$, but the two values of $B$ determined

cannot both be less than $A$, therefore we know which to choose. On the other hand, if $b$ be *greater* than $a$, $B$ must be greater than $A$, but both of the values determined may be so, therefore the solution is *ambiguous*.

42. The same results may be obtained very simply by geometrical considerations.




Let $CAB$ be the given angle, $AC$ the given side; with centre $C$ and distance $a$ (the value of the other given side) describe an arc of a circle, which, if $a$ be less than $b$ will, (as in the figure) cut the straight line $AB$ in two points $B$, $B'$ *on the same side of* $A$. Now each of the triangles $CAB$, $CAB'$, has all the data of the question, and therefore the solution is ambiguous. If $a$ had been greater than $b$, the points $BB'$ would have been on opposite sides of $A$, and there would have been only one triangle answering the given conditions.

43. It may perhaps be not without use to give a third investigation of *the ambiguous case* in the solution of triangles.

We have the formula,

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

$$\text{or } c^2 - 2bc \cos A = a^2 - b^2.$$

In this equation $a$, $b$ and $A$ are given, and we may therefore find $c$; completing the square,

$$c^2 - 2bc \cos A + b^2 \cos^2 A = a^2 - b^2 \sin^2 A;$$

$$\therefore\ c = b \cos A \pm \sqrt{a^2 - b^2 \sin^2 A}.$$

We have here *two* values of $c$, and if both values are admissible the solution is ambiguous; but the only thing which can limit their admissibility is their *sign;* hence the solution is ambiguous if both values of $c$ are *positive*,

$$\text{i.e. if } b \cos A > \sqrt{a^2 - b^2 \sin^2 A},$$

$$\text{or } b^2 \cos^2 A > a^2 - b^2 \sin^2 A,$$

$$\text{or } b^2 (\cos^2 A + \sin^2 A) > a^2;$$

$$\text{or } b > a;$$

which is the same conclusion as that at which we have arrived before.

44. *Let all the sides be given.*

This case may be solved by any of the formulæ,

$$\sin\frac{A}{2} = \sqrt{\frac{(S-b)(S-c)}{bc}},$$

$$\cos\frac{A}{2} = \sqrt{\frac{S(S-a)}{bc}},$$

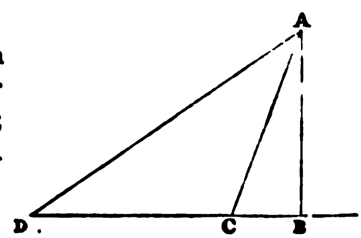
$$\tan\frac{A}{2} = \sqrt{\frac{(S-b)(S-c)}{S(S-a)}},$$

$$\sin A = \frac{2}{bc}\sqrt{S(S-a)(S-b)(S-c)}.$$

45. We have now considered all the cases of oblique-angled triangles. In practice, the triangles to be solved are frequently right-angled, in which case the solution is much simplified. One application of the methods of solving triangles is to the finding of the heights and distances of inaccessible objects; in problems of this kind we suppose the magnitudes of certain lines and angles to be measured by means of proper instruments, a description of which however would not be appropriate here. We subjoin a few simple examples of the method of

## FINDING HEIGHTS AND DISTANCES.

Ex. 1. A river of unknown breadth runs between an observer and a tower on its opposite bank; find the breadth of the river and the height of the tower.



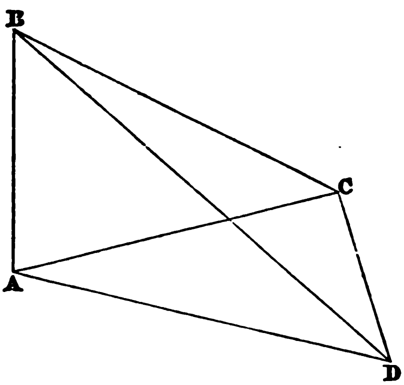
Let $AB$ be the tower, $BC$ the breadth of the river; let the angle $BCA$ be observed, and then let the observer retreat in the direction of the line $BC$ to $D$, measuring the distance $CD$, and observing the angle $CDA$.

$$\text{Let } ACB = \alpha,\ CDA = \beta,\ CD = a,$$
$$AB = x,\quad BC = y.$$
$$\text{Then } y = x \cot \alpha \text{ from the triangle } ABC,$$
$$\text{and } y + a = x \cot \beta \ldots\ldots\ldots\ldots\ldots\ldots\ldots ABD,$$

subtracting the first of these equations from the second

$$a = x(\cot \beta - \cot \alpha);$$
$$\therefore x = \frac{a}{\cot \beta - \cot \alpha} = a \frac{\sin \alpha \sin \beta}{\sin(\alpha - \beta)},$$
$$y = x \cot \alpha = a \frac{\cos \alpha \sin \beta}{\sin(\alpha - \beta)}.$$

Ex. 2. From the top of a tower, a person observes the angle of depression of two distant points in the horizontal plane, the distance of which from each other he knows, and also the angle subtended at his eye by the line joining the two points; required the height of the tower.




Let $AB$ be the tower, $C$, $D$ the two points; then the angles observed are $BCA$, $BDA$, $CBD$, which call $\alpha$, $\beta$, $\gamma$ respectively; also let $CD = a$, and let the height of the tower $= x$.

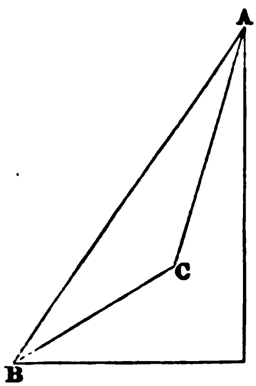
Then in the triangle $BCD$, $BC = x \operatorname{cosec} \alpha$, $BD = x \operatorname{cosec} \beta$;

$$\therefore a^2 = x^2 \operatorname{cosec}^2 \alpha + x^2 \operatorname{cosec}^2 \beta - 2x^2 \operatorname{cosec} \alpha \operatorname{cosec} \beta \cos \gamma,$$

and $x = \dfrac{a}{\sqrt{\operatorname{cosec}^2 \alpha + \operatorname{cosec}^2 \beta - 2 \operatorname{cosec} \alpha \operatorname{cosec} \beta \cos \gamma}}.$

Ex. 3. From a station $B$ at the base of a mountain, its summit $A$ is seen at an elevation of $60°$; after walking one mile towards the summit up a plane making an angle of $30°$ with the horizon, to another station $C$, the angle $BCA$ is observed to be $135°$. Find the height of the mountain.

From the triangle $ABC$, we have




$$AB = BC \cdot \frac{\sin BCA}{\sin BAC},$$

$$= BC \cdot \frac{\sin BCA}{\sin (BCA + ABC)}$$

$$= BC \cdot \frac{\sin 135°}{\sin (135° + 30°)}$$

$$= BC \cdot \frac{\sin 45°}{\sin 15°},$$

and the height of the mountain $= AB \sin 60°$

$$= BC \cdot \frac{\sin 60° \sin 45°}{\sin 15°}.$$

But $\sin 60° = \dfrac{\sqrt{3}}{2}$, $\sin 45° = \dfrac{1}{\sqrt{2}}$,

$$\sin 15° = \sin (45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30°$$

$$= \frac{\sqrt{3} - 1}{2\sqrt{2}},$$

and $BC = 1760$ yards;

$$\therefore \text{ the height} = 1760 \cdot \frac{\sqrt{3}}{\sqrt{3} - 1} = 880 (3 + \sqrt{3}),$$

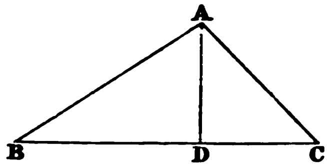
$$= 880 \times 4.7 \text{ nearly} = 4136 \text{ yards.}$$

We shall here subjoin a few propositions relating to triangles.

46. *To find the area of a triangle in terms of the sides.*

Let $ABC$ be the triangle; from $A$ draw the perpendicular $AD$ on the opposite side $BC$. Then

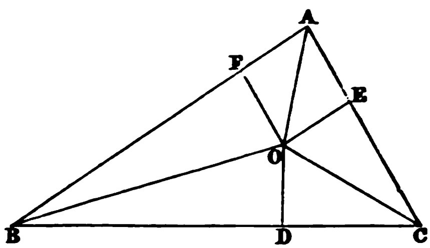
$$\text{the area} = \frac{BC \times AD}{2} = \frac{a \times b \sin C}{2};$$



but (by Art. 32) $\sin C = \frac{2}{ab}\sqrt{S(S-a)(S-b)(S-c)}$;

therefore the area $= \sqrt{S(S-a)(S-b)(S-c)}$.

47. *To find the radius of a circle inscribed in a triangle.*



Let $ABC$ be the triangle; bisect the angles, and let $O$ be the point in which the bisecting lines meet, then (by Euclid, IV. 4), $O$ is the centre, and if we draw $OD$, $OE$, $OF$, perpendicular to the sides $BC$, $AC$, $AB$, respectively, any one of these will be the radius of the inscribed circle. Call the radius $r$; then

$$\text{area of } \Delta\, ABC = \Delta\, BOC + \Delta\, AOC + \Delta\, AOB,$$

$$\text{or } \sqrt{S(S-a)(S-b)(S-c)} = \frac{ra}{2} + \frac{rb}{2} + \frac{rc}{2} = rS;$$

$$\therefore r = \sqrt{\frac{(S-a)(S-b)(S-c)}{S}}.$$

In like manner it will be found that if a circle be described touching $BC$ and $AB$, $AC$, produced, its radius will be

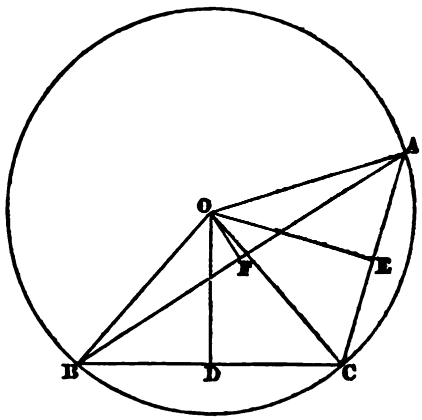
$$\frac{\sqrt{S(S-a)(S-b)(S-c)}}{S-a};$$

and similar expressions will hold for the radii of the other two circles which can be described, touching one side of the triangle and the other two produced. These three circles are sometimes called the *escribed* circles.

48. *To find the radius of a circle circumscribed about a triangle.*

Bisect the sides in the points $D$, $E$, $F$, and from the points of bisection draw perpendiculars meeting in the point $O$; then (Euc. IV. 5), $O$ is the centre of the circumscribed circle; join $OA$, $OB$, $OC$, and describe the circle $BCA$. Call the radius $R$; then (as in Art. 46)

$$\text{area of } ABC = \frac{AB \cdot BC}{2} \sin ABC,$$



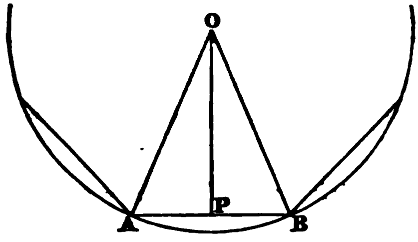

$$\text{or } \sqrt{S(S-a)(S-b)(S-c)} = \frac{ac}{2} \sin \frac{AOC}{2}, \text{ (Euc. III. 20)},$$

$$= \frac{ac}{2} \sin AOE = \frac{ac}{2} \cdot \frac{b}{2R};$$

$$\therefore R = \frac{abc}{4\sqrt{S(S-a)(S-b)(S-c)}}.$$

49. *To find the circumference and area of a regular polygon inscribed in a circle.*

Let $O$ be the centre of the circle, $AB$ one of the sides of the polygon, of which suppose that the number is $n$;



and let the radius of the circle be $r$. Join $OA$, $OB$ and draw $OP$ perpendicular to $AB$; then the angle $AOB = \dfrac{360^\circ}{n}$, and circumference of polygon $= n \,.\, AB$,

$$= 2n \,.\, AP = 2n \,.\, AO \sin AOP = 2nr \sin \frac{180^\circ}{n}.$$

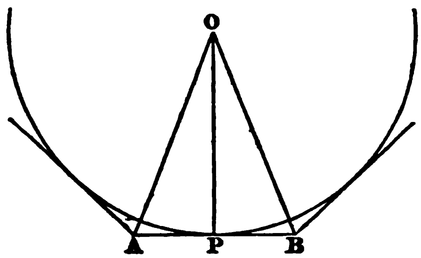
Again, area of polygon $= n$ . area of $AOB$, $= n \dfrac{AB \,.\, OP}{2}$,

$$= n \,.\, AO \sin AOP \times AO \cos AOP = nr^2 \sin \frac{180^\circ}{n} \cos \frac{180^\circ}{n}.$$

From these expressions we see, that if the number of sides is given, the *circumference* of the polygon is proportional to the *radius*, and the *area* to the *square of the radius* of the circumscribing circle.

50. *To find the circumference and area of a regular polygon circumscribed about a circle.*

Let $AB$ be any one of the sides of the polygon, touching



the circle at $P$, $n$ the number of sides, and $r$ the radius, as before. Join $OP$. Then circumference of polygon

$$= n \, . \, AB = 2n \, . \, AP = 2n \, . \, OP \tan AOP = 2nr \tan \frac{180^\circ}{n}.$$

$$\text{Again, area of polygon} = n \, . \text{ area of } AOB = n \, \frac{AB \, . \, OP}{2},$$

$$= n \, . \, OP \tan AOP \times OP = nr^2 \tan \frac{180^\circ}{n}.$$

As in the case of the inscribed polygon, we observe that the *circumference* is proportional to the *radius*, and the *area* to the *square of the radius*, the number of sides being given.

51. If we suppose a regular polygon to be inscribed in a circle, and another of the same number of sides to be circumscribed about it, it is easy to see that the greater the number of sides the more nearly will each of the polygons approximate to the other and to the circle which lies between them. In fact, suppose $C$, $C'$, to be the circumferences of the two polygons, then we have seen that

$$C = 2nr \sin \frac{180^\circ}{n},$$

$$C' = 2nr \tan \frac{180^\circ}{n};$$

$$\therefore \frac{C}{C'} = \cos \frac{180^\circ}{n}.$$

Suppose $n$ to be indefinitely great, then $\cos \frac{180^\circ}{n}$ becomes $\cos 0^\circ$ or 1, and $C = C'$.

Or let $A$, $A'$ be the areas of the two polygons, then

$$A = nr^2 \sin \frac{180^\circ}{n} \cos \frac{180^\circ}{n},$$

$$A' = nr^2 \tan \frac{180^\circ}{n};$$

$$\therefore \frac{A}{A'} = \cos^2 \frac{180^\circ}{n},$$

and if $n$ be indefinitely great, $A = A'$.

Consequently, if we suppose the number of the sides to be indefinitely increased, the inscribed and circumscribed polygon will coincide with each other, and therefore with the circle. We may therefore consider a circle as being a regular polygon having an indefinite number of sides, and may extend to it those properties which we have proved concerning polygons. Now we have seen that the circumferences of polygons are proportional to the radii, and the areas to the squares of the radii of the circles, whether inscribed or circumscribed, and therefore we conclude that the *circumferences* of circles are proportional to their *radii*, and the areas to the *squares of the radii*.

We have in fact, (taking the inscribed polygon,)

$$C = 2nr \sin \frac{180^\circ}{n}, \text{ and } A = nr^2 \sin \frac{180^\circ}{n} \cos \frac{180^\circ}{n};$$

if $n$ be indefinitely great, $\cos \frac{180^\circ}{n} = 1$, and

$$C = 2\left(n \sin \frac{180^\circ}{n}\right) r, \quad A = \left(n \sin \frac{180^\circ}{n}\right) r^2.$$

What will be the value of $n \sin \frac{180^\circ}{n}$, when $n$ is made indefinitely great? Its value, which is always denoted by $\pi$, might be calculated as follows:

In general $\cos \frac{\theta}{2} = \frac{1}{\sqrt{2}} \sqrt{1 + \cos \theta}$;

$$\therefore \cos \frac{90^\circ}{2} = \frac{1}{\sqrt{2}}, \text{ since } \cos 90^\circ = 0,$$

$$\cos \frac{90^\circ}{2^2} = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{1}{\sqrt{2}}},$$

$$\cos\frac{90^\circ}{2^3} = \frac{1}{\sqrt{2}}\sqrt{1+\frac{1}{\sqrt{2}}\sqrt{1+\frac{1}{\sqrt{2}}}},$$

&c. = &c.

By this means we can calculate the value of $\cos\frac{90^\circ}{2^p}$, where $p$ is as large a number as we please. Suppose now that $n = 2^p$, where $p$ is very large,

$$\text{then } \pi = n\sin\frac{180^\circ}{n} = 2^{p+1}\cos\frac{90^\circ}{2^p}\sin\frac{90^\circ}{2^p}$$

$$= 2^{p+1}\cos\frac{90^\circ}{2^p}\sqrt{1-\cos^2\frac{90^\circ}{2^p}}\text{, nearly.}$$

But this and other operose methods are superseded by modes of calculation of a more refined character, the introduction of which however would be unsuitable to the design of the present treatise: the result is that

$$\pi = 3.14159265 35 \ldots\ldots\ldots\ldots$$

The quantity $\pi$ may be calculated to any degree of accuracy, but it is of the class of quantities called *incommensurable*, that is, it cannot be expressed by the ratio of any two whole numbers however great: in general we may consider 3.14159 as a sufficiently accurate value of $\pi$.

The fraction $\frac{22}{7}$*, or still more nearly $\frac{355}{113}$, are approximations to the value of $\pi$.

According to the notation we have adopted,

the circumference of a circle $= 2\pi r$,

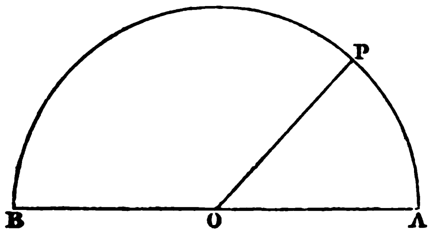
... area ............ $= \pi r^2$.

52. The introduction of the quantity $\pi$ renders this a proper place for explaining another mode of measuring angles, besides that which has been hitherto used.

* This is the value usually taken to represent $\pi$ for practical purposes in machinery. The value is too large, but it agrees with the true as far as two places of decimals.

Hitherto we have considered the right angle to be divided into 90 degrees, and have measured angles by the number of degrees they contain; but there is another mode depending upon the proposition (Euc. VI. 33) that angles at the centre of a circle are proportional to the arcs on which they stand, and which is of frequent use.

Let $POA$ be an angle at the centre $O$ of a circle, the




radius of which is $r$; $APB$ a semicircle $= \pi r$; also let the length of the arc $AP = a$. Then, by Euclid,

$$\frac{\text{angle } POA}{2 \text{ right angles}} = \frac{a}{\pi r};$$

$$\therefore \text{angle } POA = \frac{2 \text{ right angles}}{\pi} \cdot \frac{a}{r} \ldots (\text{A}).$$

Now supposing $a$ and $r$ to be given, although the angle $POA$ will be determined, yet its *numerical value* will not be settled, unless we make some convention as to what angle we shall call unity. We are at liberty to make any convention that we please, but we shall be guided in our choice by the consideration of what will make the equation (A) the most simple, and it is manifest that the most simple form will be given to that equation by making

$$\frac{2 \text{ right angles}}{\pi} = 1 \ldots\ldots\ldots\ldots\ldots (\text{B}),$$

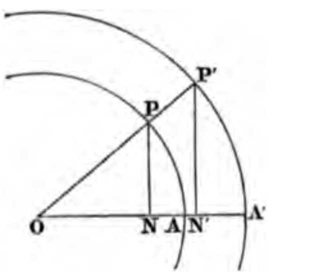
we shall then have, (denoting the numerical value of the angle $POA$ by $\theta$),

$$\theta = \frac{a}{r} \ldots\ldots\ldots\ldots\ldots\ldots (\text{C}).$$

Let us consider the results of the assumption (B). The numerical value of two right angles is the quantity $\pi$, instead of 180°, as in the former method, and the unit of angle, instead of being the ninetieth part of a right angle, is $\frac{2 \text{ right angles}}{\pi}$ or $57^\circ\ 17'\ 44''\ 48'''$ nearly*.

Again, making $\theta = 1$ in equation (C), we have $a = r$; which shews that *the unit of angle is that angle which is subtended by an arc of length equal to radius.*

53. Another mode of considering this subject is the following. Let $POA$ be any angle, and about $O$ as centre suppose any two circles described; let $PA$, $P'A'$ be the subtending arcs in the two circles, and draw $PN$, $P'N'$ perpendicular to $OAA'$; then, if the radius of the circle were given, the arc $PA$ would be a proper measure of the angle, and we might define $PA$ to be the arc, $PN$ the sine, $ON$ the cosine, &c. of the angle $POA$; but this being inconvenient, in consequence of its being necessary to know the radius, have defined $\frac{PN}{PO}$ to be the sine, $\frac{ON}{PO}$ to be the cosine, &c. of $POA$, these ratios being independent of the radius, since $\frac{PN}{PO} = \frac{P'N'}{P'O}$ and $\frac{ON}{PO} = \frac{ON'}{P'O}$; and on the same principle we should take as the measure of the angle, not the arc $PA$, but $\frac{PA}{PO}$. Thus we are led, in a rather different way from that pursued in the last article, to choose $\frac{\text{arc}}{\text{radius}}$ as the




* Prof. De Morgan makes the following remark which I gladly adopt: "the student must remember not to confound $2\pi$ with 360, as is sometimes done, even by writers. That $2\pi = 360$ is true in a certain sense; and so is $20 = 1$, for 20 *shillings* are 1 *pound*."

measure of an angle, and this choice implies that the numerical value of two right angles is $\pi$, which was our first assumption in the other case.

54. From the equation (C) (Art. 52), we see, that if $r=1$, $\theta=a$; that is to say, if we suppose the radius of the circle to be unity, the numerical value of the angle and of the subtending arc are the same. Hence, if we make this supposition respecting the radius, we are not under the necessity of making any distinction between *arcs* and *angles*, since their numerical value are the same.

55. It is frequently a matter of indifference which mode of measuring angles we adopt; but this must be carefully borne in mind, that in every example either the one or the other must be used exclusively. It will perhaps be found generally advantageous to use that last explained, or the *circular measure* as it is sometimes called*, as being the more brief.

It is easy to pass from one mode of measurement to the other: for suppose that $\theta$ is the circular measure of an angle, then the angle contains $\frac{\theta}{\pi}180$ degrees; and, conversely, if an angle contains $n^0$, its circular measure is $\frac{n}{180}\pi$.

Ex. 1. Find the circular measure of $25^0 30$.

The circular measure $= \frac{25.5}{180} \times 3.14159$

$= .445058.$

Ex. 2. Find the number of degrees, minutes, and seconds in the angle of which the circular measure is $2.5$.

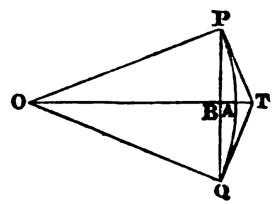
In this case

$$\frac{2.5}{3.14159} \times 180^0 = 143^0\,14'\,20'',$$

the number of degrees, minutes, and seconds required.

* Prof. De Morgan distinguishes the two methods of measuring angles as respectively the *grādual* and the *arcual* method.

56. We shall conclude this treatise with a proposition of frequent application in Mathematics, which will however require some explanation. The proposition is this, that the ratio $\frac{\sin\theta}{\theta}$, when $\theta$ is expressed in the circular measure, is indefinitely nearly equal to unity when $\theta$ is indefinitely small. This is sometimes expressed by saying that the *limiting value* of the ratio is unity, when $\theta = 0$. We are thus entering upon a subject which will be more fully discussed afterwards, but which we have already touched upon more than once; the meaning of the proposition at present under consideration will appear most clearly from the method of demonstration.




Let $PQ$ be any arc of a circle having $O$ for its centre; join $PQ$, and draw the tangents $PT$, $QT$, and let $OT$ intersect the arc $PQ$ and the chord $PQ$ in $A$ and $B$ respectively. Then it is manifest that

$$PAQ \text{ is } > PBQ \text{ and } < PT + QT,$$

$$\text{or that } PA > PB \text{ and } < PT;$$

$$\text{or if } POA = \theta, \text{ then } \theta > \sin\theta \text{ and } < \tan\theta.$$

Hence on the one hand, $\frac{\sin\theta}{\theta} < 1$,

on the other, $\frac{\tan\theta}{\theta} > 1$, or $\frac{\sin\theta}{\theta} > \cos\theta > 1 - 2\sin^2\frac{\theta}{2}$.

We have here therefore two limits between which $\frac{\sin\theta}{\theta}$ always lies, namely, between $1$ and $1 - 2\sin^2\frac{\theta}{2}$; and the smaller we make $\theta$ the more closely do these limits approach each other, and if $\theta$ be indefinitely small they approach indefinitely nearly together; in other words the limiting value of $\frac{\sin\theta}{\theta}$ is unity.

57. Hence when the angle is small the value of the sine is approximately the same as the circular measure of the angle, and the error in putting one for the other can never exceed $2\theta \sin^2 \dfrac{\theta}{2}$.

Also $\text{vers}\,\theta = 1 - \cos\theta = 2\sin^2\dfrac{\theta}{2}$,

$$\therefore \frac{\text{vers}\,\theta}{\theta^2} = \frac{1}{2}\left\{\frac{\sin\dfrac{\theta}{2}}{\dfrac{\theta}{2}}\right\} = \frac{1}{2},$$

when $\theta$ is indefinitely small.

Hence if the arc be very small, $AB \propto AP^2$.

Therefore also the difference between $OP$ and $OB$, or $AB$, is a quantity such that its ratio to the arc is indefinitely small when the arc is indefinitely small; and if the arc be a small quantity, $AB$ may be called a small quantity of the second order.

As a further example, we may shew that when $\theta$ is indefinitely small, $AT = AB$.

For $\dfrac{AT}{AB} = \dfrac{OT - OA}{OA - OB} = \dfrac{\sec\theta - 1}{1 - \cos\theta}$

$$= \frac{1}{\cos\theta} = 1, \text{ when } \theta = 0.$$

Also

$$AT - AB = AT(1 - \cos\theta) = 2AT\sin^2\frac{\theta}{2},$$

and therefore if the arc be a small quantity of the first order, $AT$ will differ from $AB$ by a quantity of the fourth order, that is, by a quantity which varies as $\theta^4$.

58. We will conclude this treatise by proving that

$$\sin\theta = \theta\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\ldots \text{ ad infinitum.}$$

We have $\sin\theta = 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}$,

$$= 2^2\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\sin\frac{\theta}{2^2}, \text{ by the same formula;}$$

and in like manner it will appear, that whatever be $n$

$$\sin\theta = 2^n\sin\frac{\theta}{2^n}\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\ldots\cos\frac{\theta}{2^n}.$$

Now let $n$ become indefinitely great, then by the preceding proposition $2^n\sin\frac{\theta}{2^n}$ becomes equal to $\theta$, and therefore

$$\sin\theta = \theta\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\ldots \text{ ad infinitum.}$$

It will be understood, that in this as in the two preceding articles the angle $\theta$ is expressed according to the circular measure.

# CONIC SECTIONS.

# CONIC SECTIONS.

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Definitions. A *right cone* is a surface generated by an indefinite straight line, which always passes through a given point, and makes a given angle with a given straight line passing through that point.

The point through which the generating line always passes, is called the *vertex* of the cone. The given straight line passing through the vertex, is called the *axis* of the cone.

The common notion of a cone is that of a pyramid standing on a circular base*; it is clear that a cone as above defined will consist of two such pyramids of indefinite height, having their axes in the same straight line and their vertices coincident.

If we conceive a cone to be cut by a plane, the curve formed by the intersection will be different according to the position of the cutting plane. There are however only *three* different modes in which it is possible for the intersection to take place.


For distinctness of conception, let the annexed figure represent a cone; $B$ is the vertex, $CDBC'D'$ is the intersection of the cone by the plane of the paper; the cone is supposed to be of indefinite length both above and below $B$. Then

(1) The cutting plane may be parallel to the line $BC$ and perpendicular to the plane of the paper, in which case it will only cut one portion

* Euclid's definition of a cone is as follows, (Euc. XI. Def. 18); A cone is a solid figure described by the revolution of a right-angled triangle about one of the sides containing the right angle, which side remains fixed.

of the cone as $BCD$, and not the other $BC'D'$, and the curve formed by the intersection will evidently be a curve of one branch and unlimited in extent, since the cone is supposed to be unlimited. This curve is called the *parabola*.

(2) The cutting plane may be inclined to the line $BC$, and may cut the cone wholly on one side of $B$, that is, may cut the portion $BCD$ without cutting the portion $BC'D'$; in this case the curve will be one of limited extent, and of an oval form. This is the *ellipse*.

(3) The cutting plane may, as in the last case, be inclined to $BC$, but may cut the cone on both sides of $B$, that is, may cut the portion $BC'D'$ as well as $BCD$; in this case the curve will consist of two branches, each of unlimited extent. This is the *hyperbola*.

In these three positions of the cutting plane are included two cases, which perhaps deserve separate notice; namely, that in which the cutting plane is perpendicular to the axis of the cone and the section consequently a circle, and that in which the plane passes through the vertex and the section is two straight lines; but these positions of the cutting plane need not be further alluded to, because the circle may be considered as a particular case of an ellipse, and the two straight lines of an hyperbola.

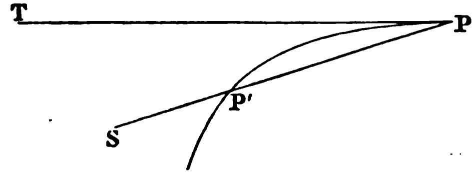
We may say therefore that there are only three different sections of a cone, the *parabola*, the *ellipse*, and the *hyperbola*, and it will be our business to study the properties of these CONIC SECTIONS in order.

It may be remarked, by the way, that the Conic Sections are curves of especial interest for three reasons; first, on account of the simplicity and elegance of their properties; secondly, because of their historical interest as curves known and studied with success by the ancients*; and, thirdly, because science has taught us that they are what may be called *physical* curves. A stone when projected describes a parabola, the planets move in ellipses, many comets describe parabolas, some hyperbolas.

* The most ancient treatise on the subject extant is that of Apollonius, of Perga in Pamphylia, who flourished in the reign of Ptolemy Euergetes, about B. C. 240. The books of Conic Sections are the only one of his works which has come down to us.

Although we have spoken of the Conic Sections as the sections of a cone, which is a mode of proceeding rendered appropriate by the name usually given to the three curves in question, we shall find it convenient in treating of their properties to adopt other definitions, and we shall have to shew that the curves so defined are really conic sections according to our present notion. It is convenient to conceive of a curve as traced by a point which moves according to an assigned law; thus we may consider a circle as a curve traced by a point, which moves under the condition of being always at the same distance from a fixed point; and this is the mode of definition which we shall adopt in the case of each of the conic sections; we shall call the curves so defined by the names of the Parabola, Ellipse, and Hyperbola, and afterwards prove that the curves defined are the three sections of a cone.

As we shall have much to do with the tangents to the conic sections, we will here explain the proper notion of the tangent of a curve. The definition given by Euclid of the tangent of a circle, namely, that it is a straight line which meets the circle and being produced does not cut it, may be taken also as the definition of a tangent in the case of a conic section. A better definition however, and one which is applicable to all curves, may be given as follows. Let $P$ be a point in a curve, $P'$ a contiguous point, draw the *secant* $PP'S$, that is the line *cutting* the curve in $P$ and $P'$. Now suppose $P'$ to approach $P$, then, when $P'$ and $P$ are *indefinitely* near together, the *secant* $SP'P$ will become the *tangent* $TP$. In other words, a tangent may be conceived of as a secant, drawn through two points in the curve indefinitely near to each other.




NOTE. It will be understood, that in this subject an algebraical notation is used for the purpose of abbreviation

only. Thus $AB . CD$ will be merely a contracted form of the phrase "the rectangle under $AB$, $CD$."

Also it may be observed, that in the figures the dotted lines refer wholly to the corollaries of the propositions.

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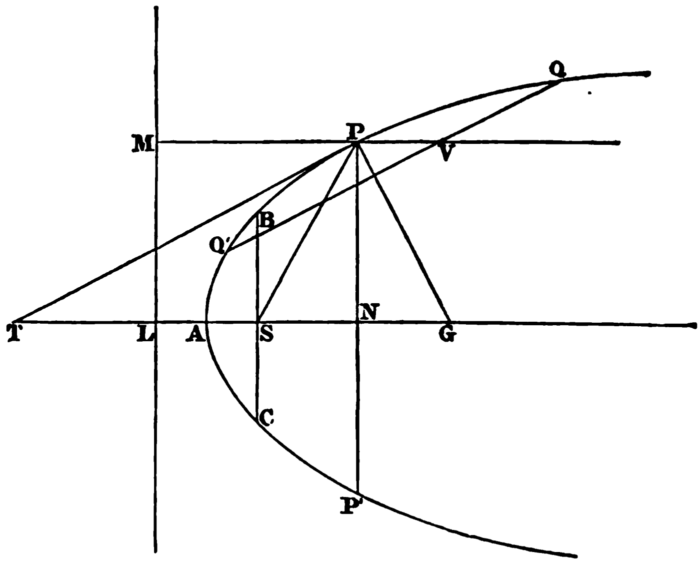
## THE PARABOLA.




Def. If a point $P$ moves in such a manner, that its distance $SP$ from a given point $S$ is always equal to its perpendicular distance $PM$ from a given fixed line $LM$, the curve traced out by $P$ is called a *parabola*.

The fixed line $LM$ is called the *directrix*, and the point $S$ the *focus*.

Draw $SL$ perpendicular to the directrix, and bisect it in the point $A$: it is manifest, from the definition, that $A$ is a point in the parabola; this point is called the *vertex*.

It is further manifest, that the curve will be exactly similar on opposite sides of the line $AS$ produced; this line is called the *axis*.

Hence it also follows, that a line $PNP'$ drawn through $P$ perpendicular to the axis to meet the parabola in $P'$ will be bisected in $N$: $PNP'$ is called an *ordinate*, and the line $AN$ an *abscissa* of the axis.

The *ordinate* $BC$, through the focus, is called the *latus rectum*.

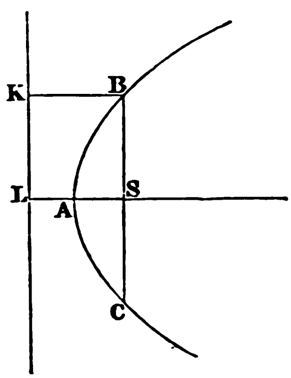
Any line $MPV$ parallel to the axis is called a *diameter*.

The names *abscissa* and *ordinate* are not confined to lines measured along the axis and perpendicular to it; they are also applied to lines measured along any diameter and parallel to the tangent at the extremity of that diameter: thus if $QVQ'$ be drawn parallel to the tangent $PT$, $PV$ is called an *abscissa of the diameter*, and $QVQ'$ an *ordinate*. The propriety of this nomenclature will be seen hereafter, when it is proved that the properties of the lines $PV$, $QVQ'$ are exactly analogous to those of $AN$, $PNP'$.

If $PT$ be a tangent to the curve at $P$, $NT$ is called the *subtangent*.

The *normal* $PG$ is a line perpendicular to the tangent. $NG$ is called the *subnormal*.

## Prop. I.




*The latus rectum* BC = 4AS.

Draw $BK$ perpendicular to the directrix, and produce $SA$ to meet the directrix in $L$; then

$$SB = BK \text{ by definition,}$$

$$= SL = 2AS,$$

since $AS = AL$ by definition;

$$\therefore BC = 2SB = 4AS.$$

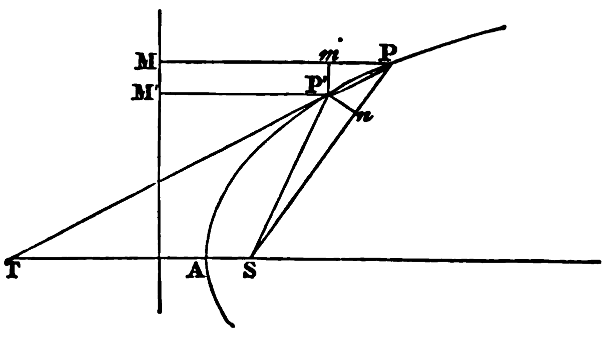
PROP. II.

*The tangent at any point of a parabola bisects the angle between the focal distance and the diameter through the point.*

Let $P$ be a point in the parabola, $P'$ a contiguous point: draw the secant $TP'P$, join $SP$, $SP'$, and draw $PM$, $P'M'$ perpendicular to the directrix, and $P'm$ perpendicular to $PM$, also in $SP$ take $Sn$ equal to $SP'$, and join $P'n$.

Then in triangles $PP'm$, $PP'n$ we have the side $PP'$ common; also

$$\begin{aligned} P'm &= PM - P'M' \\ &= SP - SP' \text{ (by definition of parabola)} \\ &= SP - Sn \text{ (by construction)} \\ &= Pn. \end{aligned}$$




Moreover since $SP' = Sn$, the angles $SP'n$, $SnP'$ are always equal, and therefore, when $P$ and $P'$ are indefinitely near together and $P'Sn$ consequently indefinitely small, each of them is a right angle. Consequently $P'nP$ is ultimately a right angle. Hence $PP'm$, $PP'n$ will be ultimately two right-angled triangles having the hypothenuse and a side of the one equal to the hypothenuse and a side of the other; and the triangles will therefore be equal in all respects.

Hence the angles $P'Pm$, $P'Pn$ are ultimately equal; that is, when the secant $TP$ becomes a tangent it bisects the angle $SPM$.*

COR. 1. The tangent at the vertex is perpendicular to the axis.

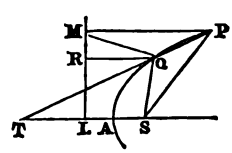
COR. 2. The normal bisects the angle between the focal distance and the diameter at the point.

COR. 3. If $T$, $G$ (see figure, page 162) are the intersections of the tangent and normal respectively with the axis, $SP = ST = SG$. (Eucl. I. 5.)

* If we define the tangent to a parabola as being a straight line, which meets the parabola and being produced does not cut it, in the place of the proposition given in the text we may substitute the following:

*The straight line, which bisects the angle between the focal distance of any point and the diameter through that point is a tangent to the parabola.*

Let $P$ be the point in the parabola; join $SP$; and draw $PM$ perpendicular to the directrix; draw $PT$ bisecting the angle $SPM$; $PT$ shall be a tangent to the parabola.




For if $PT$ meet the curve in any other point except $P$, let it meet the curve in $Q$; join $SQ$, $MQ$, and draw $QR$ perpendicular to $LM$.

Then in the triangles $MPQ$, $SPQ$, we have $MP = SP$, by definition, $PQ$ common, and the included angles $MPQ$, $SPQ$ equal by hypothesis;

$$\therefore MQ = SQ, \text{ (Euc. I. 4).}$$

Again in the triangle $MRQ$, the angle at $R$ is a right angle,

$$\therefore MQ \text{ is greater than } QR.$$

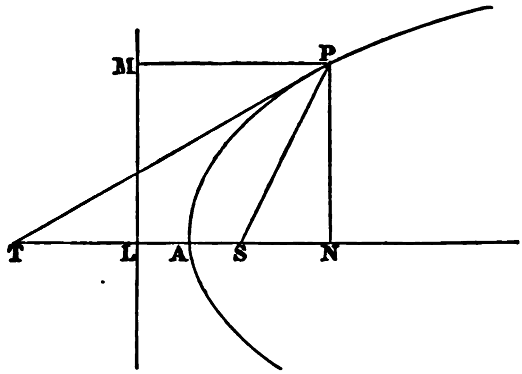
But $MQ = SQ$, $\therefore SQ$ is greater than $QR$.

But $SQ = QR$, by definition; hence $SQ$ is both equal to and greater than $QR$, which is absurd.

Hence $PT$ does not meet the parabola in any other point except $P$, therefore it is a tangent.

PROP. III.

*The subtangent is equal to twice the abscissa.* (NT = 2AN




Draw $PM$ perpendicular to the directrix $LM$; then

$$ST = SP = PM = LN,$$

$$\text{or} \quad AT + AS = AL + AN;$$

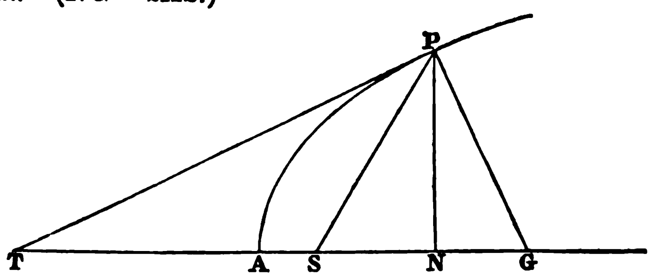
$$\text{and} \quad AS = AL;$$

$$\therefore \quad AT = AN,$$

$$\text{or} \quad NT = 2AN.$$

PROP. IV.

*The subnormal is constant, and equal to half the latu rectum.* (NG = 2AS.)




For $SG = ST = AS + AT$

$$= AS + AN \text{ (by preceding Prop.)}$$

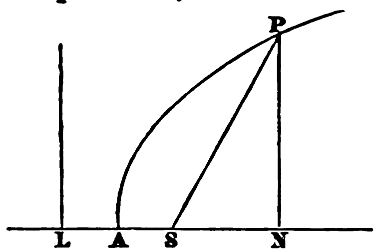
$$= 2AS + SN;$$

$$\therefore NG = SG - SN = 2AS.$$

## Prop. V.

*The rectangle under the latus rectum and the abscissa is equal to the square of a semi-ordinate of the axis.* ($PN^2 = 4AS . AN$.)

Because $AN$ is divided into two parts in $S$, if $S$ is between $A$ and $N$, or because $AS$ is divided into two parts in $N$, if $N$ is between $A$ and $S$,




$\therefore 4AS . AN + SN^2 = LN^2$ (Euc. II. 8.)

$= SP^2 = PN^2 + SN^2$;

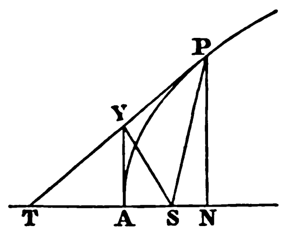
$\therefore PN^2 = 4AS . AN$.

## Prop. VI.

*If a perpendicular is drawn from the focus on the tangent the point of intersection lies in the tangent at the vertex.*

Let $AY$ be the tangent at the vertex intersecting the tangent $PT$ in $Y$; join $SY$, which shall be perpendicular to $PT$.

Because $AY$ is parallel to $PN$, and $AT = AN$; therefore $TY = PY$. (Euc. VI. 2.)




Also $SP = ST$, and $SY$ is common to the two triangles $SPY$, $STY$: therefore these two triangles have their sides respectively equal, and are therefore equal in all respects.

Therefore $\angle SYP = \angle SYT$, and therefore each is a right angle. Hence $SY$ is perpendicular to $PT$, and therefore the proposition is true.

Cor. $SY^2 = SP . AS$.

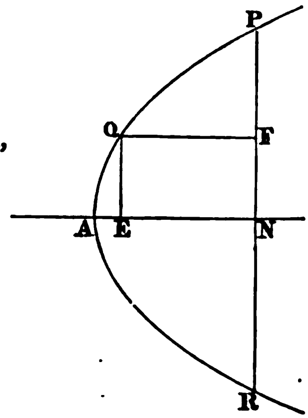
For from similar triangles $SAY$, $SYP$,

$$AS : SY :: SY : SP.$$

### Prop. VII.

*If from any point* F *in the ordinate* PR *the line* FQ *is drawn parallel to the axis and meeting the parabola in* Q, *then* PF . FR = 4AS . QF.

Draw $QE$ perpendicular to the axis; then because $PR$ is divided into two equal parts in $N$ and two unequal in $F$,




$\therefore PF . FR = PN^2 - NF^2$ (Euclid, II. 5)

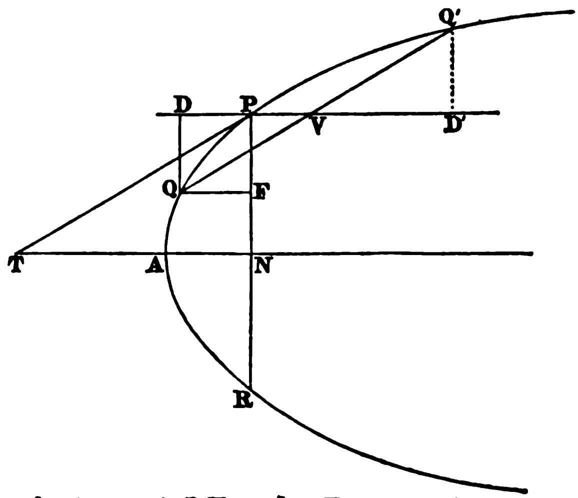
$$= PN^2 - QE^2$$

$$= 4AS . AN - 4AS . AE$$

$$= 4AS . EN = 4AS . QF.$$

### Prop. VIII.

*If from either extremity of an ordinate* QVQ′ *a perpendicular* QD *is let fall on the diameter, then* QD² = 4AS . PV.




Draw the tangent $PT$, and $QF$ perpendicular to the ordinate $PNR$, then from similar triangles $QDV$, $PNT$.

$$QD : DV :: PY : YV,$$
$$\text{but } PY^2 = 4AS \cdot AY = 2AS \cdot YV;$$
$$\therefore PY : YV :: 2AS : PY$$
$$:: 4AS : PR;$$
$$\therefore QD : DV :: 4AS : PR,$$
$$\text{or } 4AS \cdot DV = QD \cdot PR = PF \cdot PR.$$

Also (Prop. VII.)

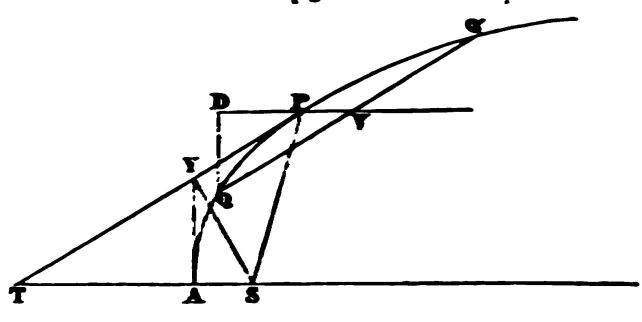
$$4AS \cdot QF = PF \cdot FR,$$
$$\text{or } 4AS \cdot DP = PF \cdot FR;$$
$$\therefore 4AS \cdot PV = PF \cdot PR - PF \cdot FR = PF^2,$$
$$\text{or } QD^2 = 4AS \cdot PV.$$

The proof would be similar if we were to draw $Q'D'$ perpendicular to the diameter from $Q'$.

COR. Hence $QD = Q'D'$, and therefore $QV = Q'V$, or a diameter bisects all its ordinates.

## PROP. IX.

*The square of a semi-ordinate of the diameter at any point is equal to four times the rectangle under the focal distance of the point and the abscissa.* ($QV^2 = 4SP \cdot PV$.)




Draw $AY$ the tangent at the vertex, $SY$ perpendicular to the tangent $PT$, $QD$ perpendicular to the diameter, and join $SP$. Then by similar triangles $QDV$, $SAY$,

$$QV^2 : QD^2 :: SY^2 : AS^2,$$

but from similar triangles, $SAY$, $SYP$,

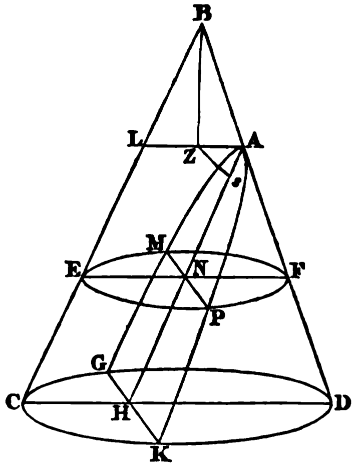
$$AS : SY :: SY : SP, \text{ or } SY^2 = AS \cdot SP;$$
$$\therefore QV^2 : QD^2 :: SP : AS,$$
$$\text{but } QD^2 = 4AS \cdot PV; \text{ (Prop. VIII.)}$$

$$\therefore QV^2 : 4AS . PV :: SP : AS,$$
$$\text{or } QV^2 = 4SP . PV.^*$$

Obs. It will be seen that this proposition includes Prop. v., since in that case $P$ coincides with $A$ and $SP = AS$.

## Prop. X.

*If a right cone is cut by a plane which is parallel to a line in its surface, and perpendicular to the plane containing that line and the axis, the section is a parabola.*

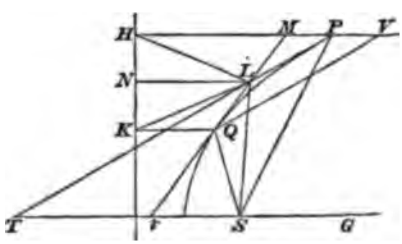



* The following is a very neat independent proof of this proposition.

Draw the tangents $PT$, $Qt$ intersecting in $L$; and $QK$, $LN$, $PMH$ perpendicular to the directrix. Join $LH$, $LK$, $SP$, $SQ$.

Then $QSP = QSG - PSG$
$= 2(StQ - STP) = 2PLM.$

Again in the triangles $HPL$, $SPL$, we have $HP$, $PL$ equal to $SP$, $PL$ respectively, and the included angles equal, therefore the triangles are equal in all respects, and $HL = SL$.




The same holds good of the triangles $KQL$, $SQL$, and therefore $KL = SL$;

$$\therefore HL = KL.$$

And since $LN$ is common to the triangles $HLN$, $KLN$ and is perpendicular to $HK$, the triangles $HNL$, $KNL$ are equal in all respects.

$$\therefore \angle PHL = \angle LKQ = \angle QSL = \angle LSP = \tfrac{1}{2} \angle QSP = \angle PLM.$$

Hence $PLM$, $PHL$ are similar triangles.

Therefore $PL^2 = PM . PH$.

But $HN = NK$, $\therefore QL = LM$, $\therefore PM = PV$, and $QV = 2PL$.

$$\therefore QV^2 = 4PL^2 = 4SP . PV.$$

Let $BCD$ be the section of the cone by the plane of the paper, $AGK$ the cutting plane which is supposed perpendicular to the plane of the paper and parallel to $BC$. Let $EMFP$ be any circular section made by a plane perpendicular to the axis of the cone. Then the line $MNP$, in which the planes $EMFP$, $AGK$ intersect, is manifestly perpendicular to both of the lines $ENF$, $ANH$, in which those planes intersect the plane of the paper. Draw $AL$ parallel to $CD$, $BZ$ perpendicular to $AL$, and $ZS$ perpendicular to $AN$.

Then by similar triangles $BLZ$, $ZAS$,

$$BL : LZ = ZA : AS,$$

$$\therefore BL : LA = ZA : 2AS,$$

$$\text{or } BL : LA = EN : 4AS.$$

Again by similar triangles $BLA$, $ANF$,

$$BL : LA = AN : NF;$$

$$\therefore EN : 4AS = AN : NF,$$

$$\text{or } EN \cdot NF = 4AS \cdot AN.$$

But since $EPF$ is a semicircle $EN \cdot NF = PN^2$,

$$\therefore PN^2 = 4AS \cdot AN,$$

which is a property of a parabola of which the focus is $S$. (Prop. v.) Hence the curve $GAK$ is a parabola.

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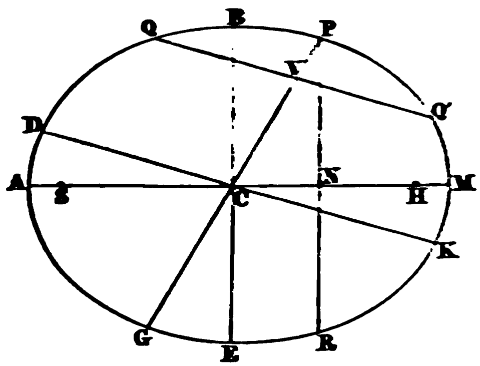
## THE ELLIPSE.




Def. If a point $P$ move in such a manner that the sum of its distances from two fixed points $S$, $H$ is always the same, the curve traced out by $P$ will be an *ellipse*.

The points $S$, $H$ are called the *foci*, and the point $C$ bisecting $SH$ the *centre*.

Any straight line $PCG$ through the centre is called a *diameter:* it is manifest that the centre bisects all such lines.

The diameter $ASHM$ through the foci is called the *axis major:* $A$, $M$ are called *vertices*.

A line $PNR$ perpendicular to the axis major is called an *ordinate*, and the lines $AN$, $NM$ *abscissæ* of the axis.

The ordinate $BCE$ through the centre is called the *axis minor*, and that through either focus the *latus rectum*.

The diameter $DCK$ which is parallel to the tangent at $P$ is said to be *conjugate* to $PCG$.

A line $QVQ'$ parallel to the conjugate diameter is called an *ordinate* of the diameter $PCG$, and the lines $PV$, $VG$ *abscissæ*.

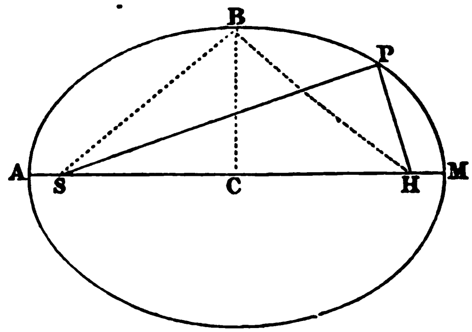
A perpendicular to the axis major through the point in which a tangent at the extremity of either latus rectum meets the axis is called the *directrix**.

## Prop. I.

*The sum of the focal distances of any point is equal to the axis major.* (SP + HP = 2AC).

For, by definition,

$$SP + HP = SA + HA,$$
$$SP + HP = SM + HM;$$




* This line is so called, because it may be proved, (see Prop. v. Cor. 2,) that if from a point $P$ in the ellipse $PM$ is drawn perpendicular to the directrix, then $SP$ is always less than $PM$ in a constant ratio. Thus the ellipse might be defined in a manner similar to that adopted for the parabola. The ratio above mentioned is called the *eccentricity* of the ellipse.

$$\therefore 2(SP+HP) = SA + SM + HA + HM$$
$$= 2AM;$$
$$\text{or } SP + HP = AM = 2AC.$$

COR. $SB + HB = 2AC.$

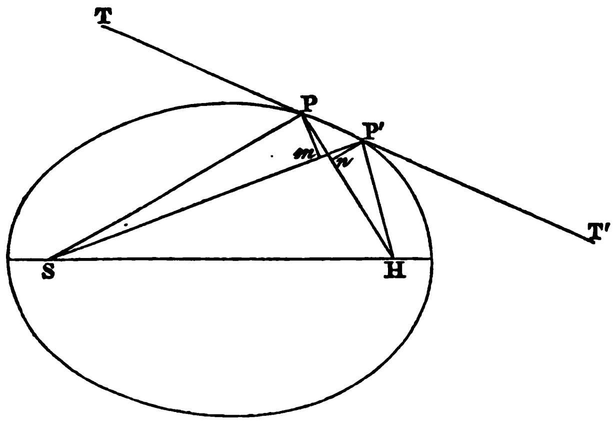
But manifestly $SB = HB$;

$$\therefore SB = AC.$$

Hence also $BC^2 = AC^2 - SC^2.$

PROP. II.

*The tangent at any point of an ellipse makes equal angles with the focal distances.*




Let $P$, $P'$ be two contiguous points in the ellipse; draw the secant $TPP'T'$; join $SP$, $SP'$, $HP$, $HP'$; in $SP'$ take $Sm$ equal $SP$, in $HP$ take $Hn$ equal to $HP'$, and join $Pm$, $P'n$.

Then in the triangles $PmP'$, $P'nP$, we shall have,

$$Pn = HP - HP',$$
$$\text{and } P'm = SP' - SP,$$
$$\text{but } SP + HP = SP' + HP',$$
$$\therefore Pn = P'm;$$

Moreover since $SP = Pm$, the angles $SPm$, $SmP$ are always equal, and therefore, $P$ and $P'$ are indefinitely near together, and $PSm$ consequently indefinitely small, each of them is a right angle. Consequently $PmP'$ is ultimately a right angle; and so is $PnP'$ for like reasons.

And therefore in the right-angled triangles $PmP'$, $PnP'$, we have the side $P'm = Pn$, and the side $PP'$ common: hence the triangles are equal in all respects, and $\angle PP'm = \angle P'Pn$: but when $P$ and $P'$ coincide these are the angles which the tangent makes with the focal distances: hence the proposition is true*.

Cor. 1. The tangent bisects the angle between $HP$ and $SP$ produced.

Cor. 2. The tangent at either vertex is perpendicular to the axis major.

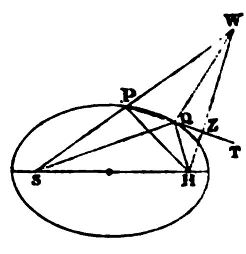
## Prop. III.

*The perpendiculars from the foci on the tangent intersect the tangent in the circumference of a circle, having the axis major as diameter.*

Produce $SP$ to $W$, making $PW = HP$: join $WH$, cutting the tangent in $Z$: join $CZ$.

* The following demonstration of the fundamental property of the tangent of an ellipse is analogous to that given at page 165 for the parabola.

*The straight line which bisects the angle between one focal distance of a point and the other focal distance produced is a tangent to the ellipse at that point.*




Let $P$ be a point in the ellipse; join $SP$, $HP$, and produce $SP$ to $W$; bisect the angle $HPW$ by the straight line $PT$; $PT$ shall touch the ellipse.

For if not, let $PT$ cut the ellipse in $Q$; from $H$ draw $HZ$ perpendicular to $PT$ and produce it to meet $SPW$ in $W$. Join $SQ$, $HQ$, $WQ$.

Then in the triangles $HPZ$, $WPZ$, we have the angles $HPZ$, $WPZ$ equal, by construction; $HZP$, $WZP$ equal, being right angles; and the side $PZ$ common; hence the triangles are equal in all respects, and $HP = PW$, and $HZ = WZ$. (Euc. I. 26.)

$$\therefore SW = SP + PW = SP + HP = 2AC \text{ (Prop. I.)}$$

Again, in the triangles $HQZ$, $WQZ$, we have $HZ$, $WZ$ equal by the preceding demonstration, $QZ$ common, and the included angles $HZQ$, $WZQ$ equal, being right angles; hence the triangles are equal in all respects, and $HQ = WQ$. (Euc. I. 4.)

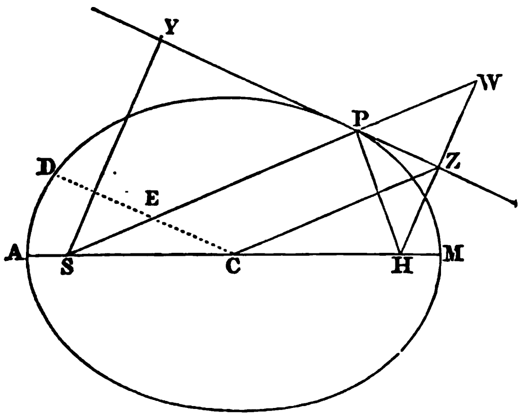
$$\therefore SQ + WQ = SQ + HQ = 2AC \text{ (Prop. I.)}$$

$$\text{but } SW = 2AC;\ \therefore SW = SQ + WQ,$$

which is impossible. (Euc. I. 20.)

Hence $PT$ does not cut the ellipse in any point such as $Q$; therefore it is a tangent.

Then in the triangles $HPZ$, $WPZ$, we have the sides $HP$, $WP$ equal by construction, $PZ$ common, and the angles $HPZ$, $WPZ$ equal by the property of the tangent: therefore the triangles are equal in all respects, and $\angle PZH = \angle PZW$, each of which is therefore a right angle. Hence $HZ$ is the perpendicular on the tangent.




Again, $SC = CH$, and $WZ = ZH$; $\therefore$ $CZ$ is parallel to $SW$; and by similar triangles $CZH$, $SWH$, $CZ = \frac{1}{2} SW$. But $SW = SP + PW = SP + PH = 2AC$; $\therefore$ $CZ = AC$, and therefore $Z$ is a point in a circle, the centre of which is $C$ and radius $AC$.

The proof would have been the same, if we had considered $SY$ the perpendicular from the focus $S$.

Cor. Draw the conjugate diameter $CD$ cutting $SP$ in $E$. Then $PECZ$ is a parallelogram;

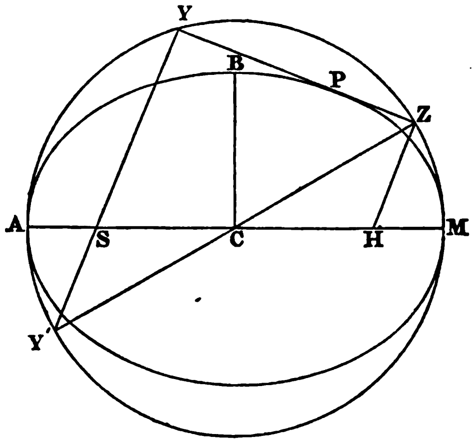
$$\therefore PE = CZ = AC.$$

Note. As the circle, which is the subject of the preceding proposition, is of great use in demonstrating the properties of the ellipse, we shall call it the *auxiliary circle*.

## Prop. IV.

*The rectangle under the perpendiculars from the foci on the tangent is equal to the square of the semi-axis minor.*

$(SY \,.\, HZ = BC^2)$.




Produce $SY$ to meet the auxiliary circle in $Y'$; and join $CY'$, $CZ$. Then because $ZYY'$ is a right angle, therefore $ZCY'$ is a diameter of the auxiliary circle, and $CZ$, $CY'$ are in the same straight line. Hence in the triangles $SCY'$, $ZCH$, the sides $SC$, $CY'$ are respectively equal to $HC$, $CZ$, and $\angle SCY' = \angle HCZ$; therefore the triangles are equal in all respects, and $SY' = HZ$.

$$\therefore SY \,.\, HZ = SY \,.\, SY' = AS \,.\, SM; \text{ (Eucl. III. 35)}$$

$$\text{but } AS \,.\, SM = AC^2 - SC^2 \text{ (Eucl. II. 5)}$$

$$= BC^2; \text{ (Prop. I. Cor.)}$$

$$\therefore SY \,.\, HZ = BC^2.$$

Cor. By similar triangles, $SYP$, $HZP$, (fig. Prop. III.)

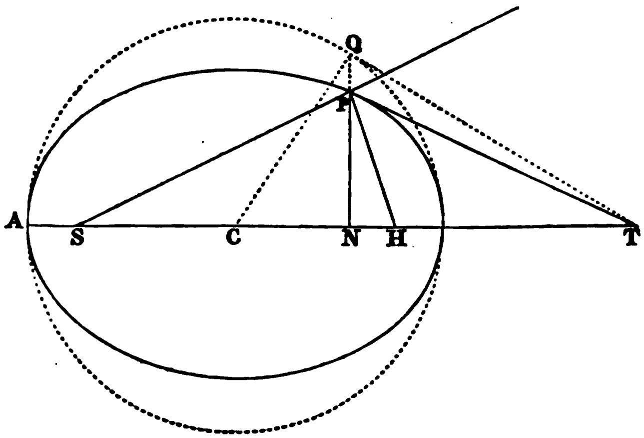
$$SY : HZ :: SP : HP;$$

$$\therefore SY^2 : SY \,.\, HZ :: SP : HP,$$

$$\therefore SY^2 : BC^2 :: SP : 2AC - SP.$$

## Prop. V.

*The rectangle under the lines intercepted between the centre and the intersections of the axis with the ordinate and tangent respectively, is equal to the square of the semi-axis major.* ($CN.CT = AC^2$.)




Produce $SP$ to $W$, then because $PT$ bisects the exterior angle $WPH$,

$$\therefore ST : HT :: SP : HP. \quad \text{(Eucl. VI. A.)}$$

$$\therefore ST + HT : ST - HT :: SP + HP : SP - HP,$$

$$\text{or } 2CT : SH :: 2AC : SP - HP.$$

Again, we have $SP^2 = SN^2 + PN^2$,

$$HP^2 = HN^2 + PN^2;$$

$$\therefore SP^2 - HP^2 = SN^2 - HN^2,$$

$$\text{or } (SP - HP)(SP + HP)$$

$$= (SN - HN)(SN + HN), \text{ (Eucl. II. 5. Cor.)}$$

$$\text{or } (SP - HP)\,2AC = 2CN.SH,$$

$$\text{or } 2CN : SP - HP :: 2AC : SH.$$

Hence $CT : AC :: AC : CN$,

$$\text{or } CN.CT = AC^2.$$

Cor. 1. Produce $PN$ to meet the auxiliary circle in $Q$. Join $CQ$, $TQ$.

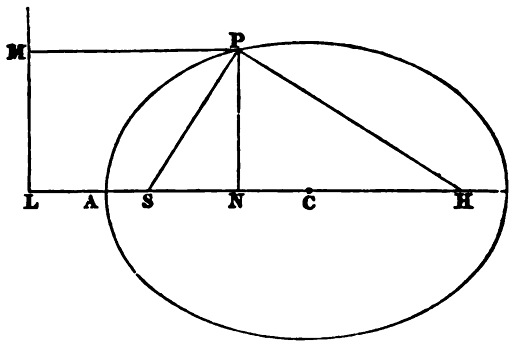
Then $CT.CN = AC^2 = CQ^2$,

$$\text{or } CT : CQ :: CQ : CN;$$

therefore $CQT$ is a right angle, and $QT$ touches the circle at $Q$; that is, *the tangents of the ellipse and circle at $P$ and $Q$ respectively cut the major-axis produced in the same point.*

COR. 2. From $P$ draw $PM$ perpendicular to the directrix $LM$. Then as in the proposition

$$(SP - HP)\,2AC = 2CN \,.\, SH,$$
$$\text{or } (2AC - 2SP)\,2AC = 2CN \,.\, 2CS,$$
$$\text{or } SP \,.\, AC = AC^2 - CN \,.\, CS.$$



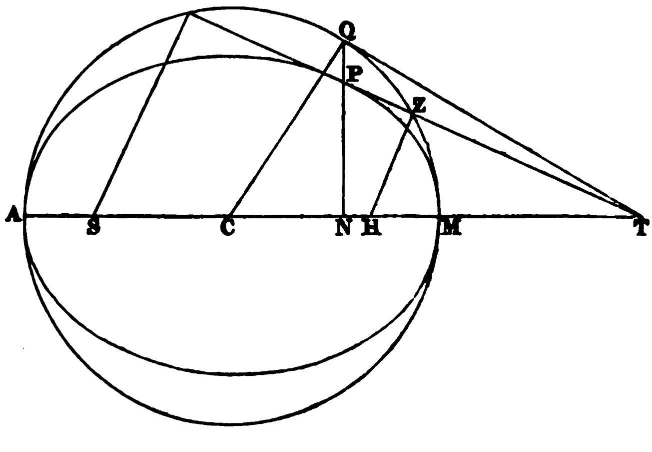

But by the definition of the directrix, and by the proposition, $AC^2 = CS \,.\, CL = CS \,.\, CN + CS \,.\, NL,$

$$\therefore SP \,.\, AC = CS \,.\, NL = CS \,.\, PM,$$
$$\text{or } SP : PM :: CS : AC,$$

in other words the ratio of $SP$ to $PM$ is the same whatever be the position in the ellipse of the point $P$.

PROP. VI.

*The rectangle under the abscissæ of the axis major is to the square of the semi-ordinate, as the square of the axis major to the square of the axis minor.* ($AN \,.\, NM : PN^2 :: AC^2 : BC^2$.)



Draw the tangent $PT$ and the perpendiculars upon it from the foci $SY$, $HZ$; produce $PN$ to meet the auxiliary circle in $Q$, join $CQ$, and draw the tangent $QT$: then the triangles $SYT$, $PNT$, $HZT$ are similar,

$$\therefore PN : SY :: NT : YT,$$

$$\text{and } PN : HZ :: NT : ZT;$$

$$\therefore PN^2 : SY . HZ :: NT^2 : YT . ZT,$$

or $PN^2 : BC^2 :: NT^2 : QT^2$ (Euc. III. 36),

$$:: QN^2 : CQ^2 \text{ by similar triangles},$$

$$:: AN . NM : CQ^2 \text{ by property of the circle},$$

$$\text{or } AN . NM : PN^2 :: AC^2 : BC^2.$$

COR. 1. Hence if $L$ be the latus rectum,

$$L . AC = 2BC^2.$$

For by the proposition

$$AS . SM : \frac{L^2}{4} :: AC^2 : BC^2,$$

or $$BC^2 : \frac{L^2}{4} :: AC^2 : BC^2, \text{ (See Prop. IV.)}$$

or $$BC : \frac{L}{2} :: AC : BC,$$

$$\therefore L . AC = 2BC^2.$$

COR. 2. $$AN . NM = QN^2;$$

$$\therefore QN : PN :: AC : BC.$$

COR. 3. Hence it may be shewn that a theorem analogous to Prop. V. holds for the minor axis; that is, if the tangent meet the axis minor in $t$, and $Pn$ be perpendicular to the axis minor, then $Cn . Ct = BC^2$.

For we have

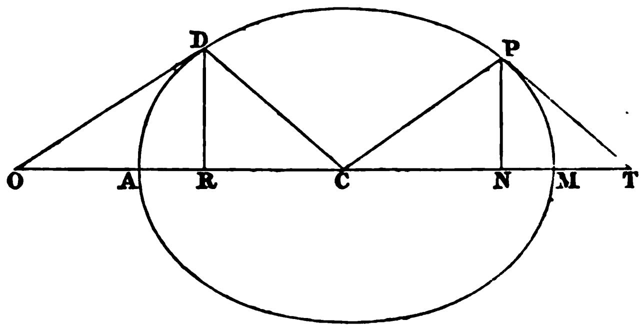
$$Ct : CT :: PN : NT,$$

$$Cn : CN :: PN : CN,$$

$$\therefore Cn . Ct : CN . CT :: PN^2 : CN . NT,$$
$$\begin{aligned} Cn . Ct : AC^2 &:: PN^2 : CN . CT - CN^2 \\ &:: PN^2 : AC^2 - CN^2 \\ &:: PN^2 : AN . NM \\ &:: BC^2 : AC^2, \end{aligned}$$
$$\therefore Cn . Ct = BC^2.$$

PROP. VII.

*If the semi-diameter* CD *is conjugate to* CP, *then* CP *is conjugate to* CD.




Draw the ordinates $PN$, $DR$ and the tangents $PT$, $DO$. Then the triangles $PNT$, $CDR$ are similar.

But $\quad CN . CT = AC^2$;

$$\begin{aligned} \therefore CN . NT &= AC^2 - CN^2 \\ &= AN . NM; \text{ (Euc. II. 5. Cor.)} \end{aligned}$$

in like manner

$$CR . RO = AR . RM;$$

$$\begin{aligned} \therefore CN . NT : CR . RO &:: AN . NM : AR . RM, \\ &:: PN^2 : DR^2, \\ &:: NT^2 : CR^2, \\ \therefore CN : RO &:: NT : CR, \\ &:: PN : DR; \end{aligned}$$

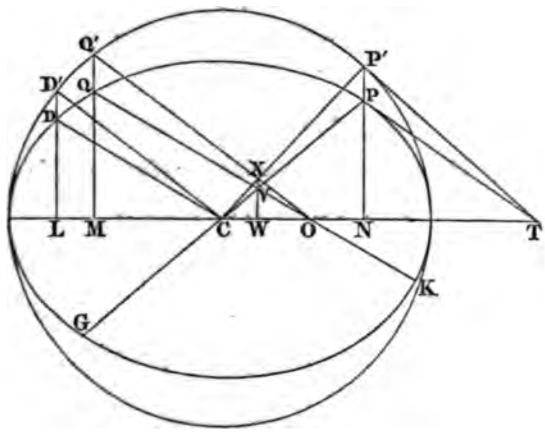
therefore the triangles $PCN$, $DOR$ are similar, and $CP$ is parallel to $DO$, or $CP$ is conjugate to $CD$.

Obs. It is evident that the major and minor axes are *conjugate diameters.*

## Prop. VIII.

*The rectangle under the abscissæ of any diameter is to the square of the semiordinate, as the square of the diameter to the square of the conjugate.*

$$(PV \cdot VG : QV^2 :: CP^2 : CD^2.)$$




Let the ordinate $QVK$ meet the axis in $O$. Produce the ordinates $NP$, $LD$ to meet the auxiliary circle in $P'$, $D'$ respectively: draw $QM$, $VW$ perpendicular to the axis: let $VW$ cut $CP'$ in $X$: join $OX$, and produce it to meet $MQ$ produced in $Q'$.

$$\text{Then } Q'M : QM :: XW : VW$$
$$:: P'N : PN;$$

therefore $Q'$ is a point in the auxiliary circle.

Again, draw the tangents $PT$, $P'T$ to the ellipse and circle, then $CD$ is parallel to $PT$.

$$\therefore CL : NT :: DL : PN :: D'L : P'N;$$

therefore the triangles $CLD'$, $TNP'$ are similar, and $CD'$ is parallel to $P'T$ and perpendicular to $CP'$.

In like manner $Q'X$ is perpendicular to $CP'$.

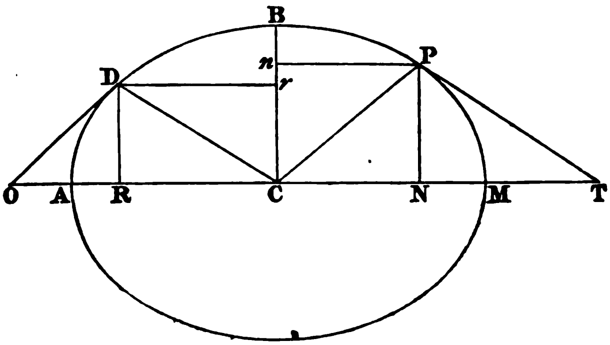
$$\text{Now } CP^2 : CV^2 :: CP'^2 : CX^2;$$
$$\therefore CP^2 - CV^2 : CP'^2 - CX^2 :: CP^2 : CP'^2,$$
$$\text{or } PV . VG : Q'X^2 :: CP^2 : AC^2;$$
$$\text{and } Q'X^2 : QV^2 :: CD'^2 : CD^2 :: AC^2 : CD^2;$$
$$\therefore PV . VG : QV^2 :: CP^2 : CD^2.$$

Obs. It will be seen that Prop. vi, is a particular case of this proposition, since in that case $CP = AC$ and $CD = BC$. It has already been remarked that the axes of the ellipse are conjugate diameters.

## Prop. IX.

*The sum of the squares of conjugate diameters is constant.*

$$(CP^2 + CD^2 = AC^2 + BC^2.)$$




Draw the tangents $PT$, $DO$, the ordinates $PN$, $DR$, and $Pn$, $Dr$ ordinates to the minor axis.

$$\text{Then } CN . CT = AC^2 = CR . CO;$$

$\therefore CN : CR :: CO : CT,$

$:: CD : PT$ by similar triangles $CDO$, $TPC$;

$:: CR : NT$ by similar triangles $CDR$, $TPN$;

$$\text{or } CR^2 = CN . NT = CN . CT - CN^2$$
$$= AC^2 - CN^2;$$
$$\text{or } CR^2 + CN^2 = AC^2.$$

Similarly, (See Prop. VI. Cor. 3) $Cn^2 + Or^2 = BC^2$;

$$\therefore\ CN^2 + Cn^2 + CR^2 + Cr^2 = AC^2 + BC^2,$$

$$\text{or } CP^2 + CD^2 = AC^2 + BC^2.$$

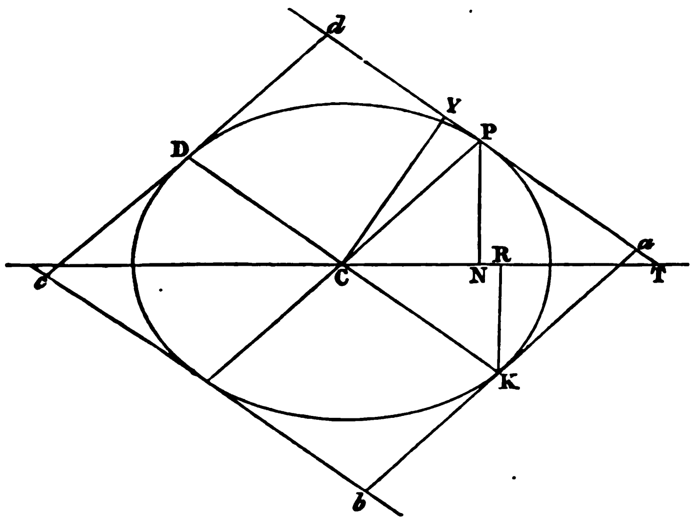
Cor. Since $CR^2 = AC^2 - CN^2$, $\therefore$ $CR$ = the ordinate in the auxiliary circle corresponding to the point $P$;

$$\therefore\ PN : CR :: BC : AC;$$

$$\text{similarly, } DR : CN :: BC : AC.$$

## Prop. X.

*Parallelograms circumscribing an ellipse, the sides of which are parallel to conjugate diameters, have the same area.*




Let $abcd$ be the parallelogram formed by tangents parallel to the semiconjugate diameters $CP$, $CD$. Draw $CY$ perpendicular to the tangent $PT$, and the ordinates $PN$, $KR$.

Then by similar triangles $CYT$, $CKR$,

$$CT : CY :: CK : KR;$$

$$\therefore\ CY \,.\, CK = CT \,.\, KR;$$

$$\text{but } CT : AC :: AC : CN,$$

$$\text{and } KR : CN :: BC : AC \text{ (Prop. IX. Cor.)}$$

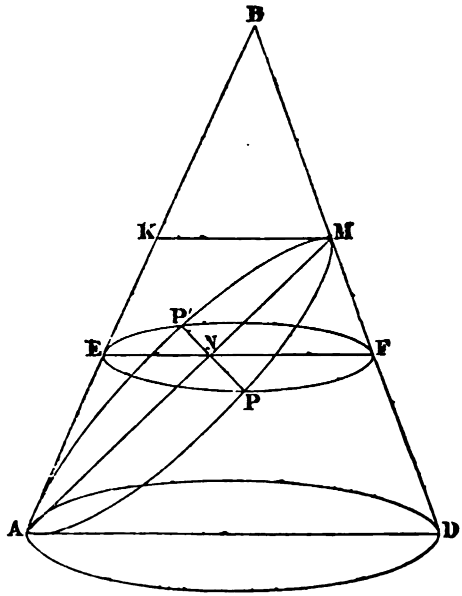
$$\therefore CT \cdot KR = AC \cdot BC,$$

$$\text{and } \therefore CY \cdot CK = AC \cdot BC;$$

but $CY \cdot CK$ is one fourth of the parallelogram *abcd*, therefore the parallelogram $abcd = 4AC \cdot BC =$ the rectangle under the axes of the ellipse.

## Prop. XI.

*If a right cone is cut by a plane which is not parallel to a line in its surface, and the section is wholly on one side of the vertex, the section is an ellipse.*




Let $BAD$ be the section of the cone by the plane of the paper, $APMP'$ the cutting plane which is supposed perpendicular to the plane of the paper. Let $EPFP'$ be any circular section made by a plane perpendicular to the axis of the cone. Then the line $PNP'$, in which the planes $EPFP'$, $APMP'$ intersect, is manifestly perpendicular to both $EF$

and $AM$. Draw $MK$ parallel to $AD$. Then by similar triangles,

$$AN : EN :: AM : KM,$$

$$\text{and } NM : NF :: AM : AD;$$

$$\therefore AN.NM : EN.NF :: AM^2 : KM.AD,$$

$$\text{or } AN.NM : PN^2 :: AM^2 : KM.AD;$$

but, by Prop. VI, this is a property of an ellipse, the major axis of which is $AM$, and the minor axis a mean proportional between $KM$ and $AD$; hence the section is an ellipse.

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## THE HYPERBOLA.

DEF. If a point $P$ move in such a manner that the difference of its distances from two fixed points $S$, $H$ is always the same, the curve traced out by $P$ will be an *hyperbola**.

The points $S$, $H$ are called the *foci*, and the point $C$ bisecting $SH$ the *centre*.

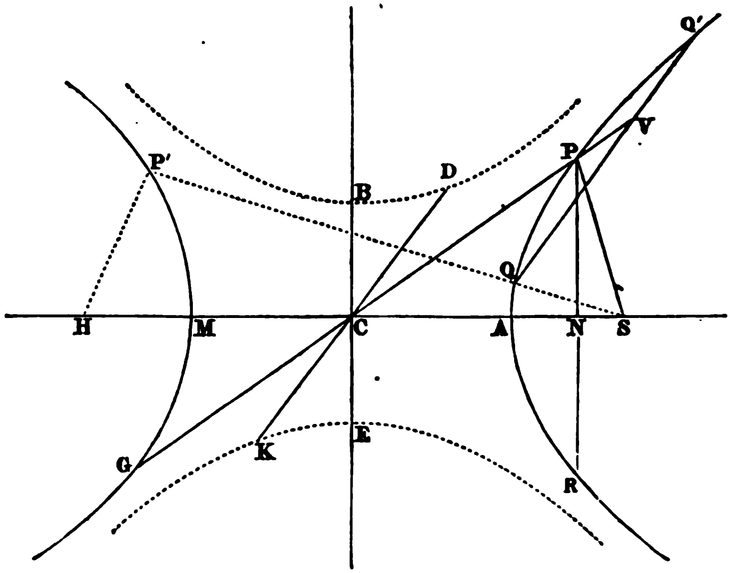
It is evident that the hyperbola must consist of *two branches*, because for every point $P$ to the right of $C$ there will be another $P'$ to the left of it, exactly similarly situated; consequently the curve will be exactly symmetrical with respect to $C$. Moreover, it is clear that the branches will be infinite, since there is no limit put to the magnitudes of $HP$ and $SP$ by the condition of their *difference* being constant.

The definition of the hyperbola being so nearly analogous to that of the ellipse, it will be anticipated that many of their

* The hyperbola might be defined as the locus of a point, the distance of which from a given point (*the focus*), is always greater in a constant ratio than its distance from a given straight line (the *directrix*). Thus the conic sections admit of a simple definition applicable to all three varieties; for we may say that a conic section is the locus of a point, the distance of which from a given point is in a constant ratio to its distance from a given line, the ratio being one of equality for the parabola, of less inequality for the ellipse, and of greater inequality for the hyperbola. (See Prop. v. Cor. 2, page 178, and Prop. v. Cor. 2, Page 192.)

properties will be possessed in common; also the nomenclature is so devised, as to preserve as much as possible the analogy between the two curves.

Any line $PCG$ through the centre and meeting the two branches of the curve is called a *diameter*: it is manifest that the centre bisects all such lines.




The diameter $ACM$, which produced passes through the foci, is called the *axis major*: $A$, $M$ are called *vertices*.

$PNR$ is an *ordinate* of the axis; $AN$, $NM$ the *abscissæ*.

A line $BC$ drawn from $C$, perpendicular to the axis major, and such that $BC^2 = SC^2 - AC^2$, is called the *semiaxis minor*; $BE$ ($= 2BC$) is the *axis minor*; this is according to the analogy of the ellipse, in which we had $BC^2 = AC^2 - SC^2$. (Ellipse, Prop. I. Cor.)

An hyperbola (represented by a dotted line in the figure), having $BE$ for its major and $AM$ for its minor axis, is called the *conjugate hyperbola*.

When the major and minor axes are equal, the hyperbola is said to be *rectangular*; it will be seen that the rectangular

has the same kind of relation to the common hyperbola, as the circle has to the ellipse.

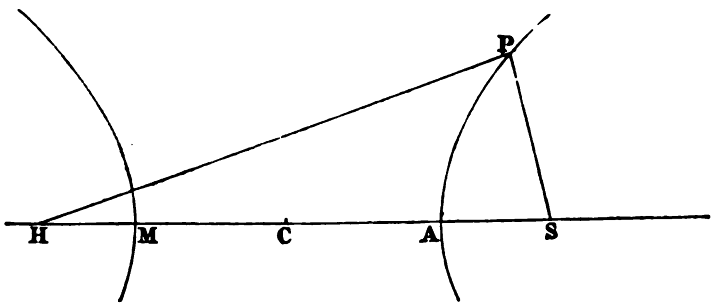
A line $KCD$ drawn through $C$, parallel to the tangent at $P$, to meet the conjugate hyperbola, is called the *diameter conjugate* to $PCG$.

Other definitions are the same as those for the ellipse.

An *asymptote* to a curve is a straight line which continually approaches the same, and the distance between which and the curve becomes less than any assignable distance, but which being produced ever so far does not cut it.

### Prop. I.

*The difference of the focal distances of any point is equal to the axis major.* (HP − SP = 2AC.)




For, by definition,

$$HP - SP = HA - SA,$$

$$HP - SP = SM - HM;$$

$$\therefore 2(HP - SP) = HA - HM + SM - SA,$$

$$= 2AM,$$

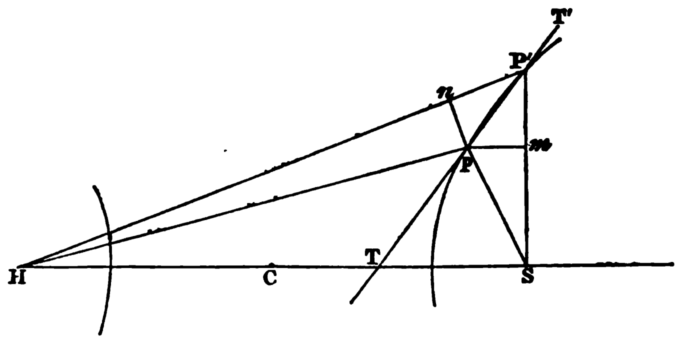
$$\text{or } HP - SP = AM = 2AC.$$

### Prop. II.

*The tangent at any point of an hyperbola makes equal angles with the focal distances.*

Let $P$, $P'$ be two contiguous points in the hyperbola; draw the secant $TPP'T'$; join $SP$, $SP'$, $HP$, $HP'$; in $SP'$

take $Sm$ equal to $SP$, and in $HP'$, $Hn$ equal to $HP$; and join $Pm$, $Pn$.




Then $$HP - SP = HP' - SP',$$
by the property of the hyperbola,
$$\therefore SP' - SP = HP' - HP,$$
$$\text{or } P'm = P'n;$$

and hence (as in the case of the ellipse) the triangles $PmP'$, $PnP'$ will be equal in all respects when $P$ and $P'$ are indefinitely near together.

Hence in this case
$$\angle PP'm = \angle PP'n:$$

but when $P$ and $P'$ coincide these are the angles which the tangent makes with the focal distances; hence the proposition is true*.

* The following is analogous to the proposition given in the note on page 174.

*The straight line which bisects the angle between the focal distances of any point is a tangent to the hyperbola at that point.*




Let $PT$ be the straight line which bisects the angle $SPH$; then if it be not a tangent to the hyperbola at $P$, let $PT$ or $PT$ produced cut the curve in another point $Q$. Join $SQ$, $HQ$; from $S$ draw $SY$ perpendicular to $PT$, and produce it to meet $HP$ in $W$; join $WQ$.

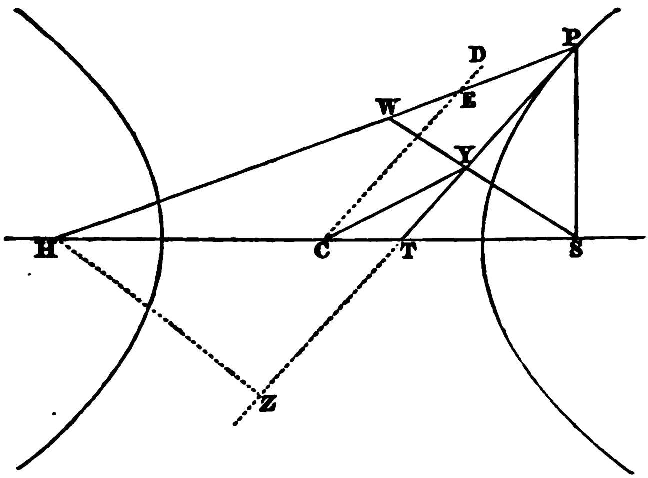
Then in the triangles $SPY$, $WPY$, we have the angles $SPY$, $WPY$ equal by hypothesis, and the angles $SYP$, $WYP$ equal, each being a right

COR. The tangent at either vertex is perpendicular to the axis major.

## PROP. III.

*The perpendiculars from the foci on the tangent intersect the tangent in the circumference of a circle having the axis major as diameter.*

In $HP$ take $PW = SP$; join $SW$, cutting the tangent $PT$ in $Y$: join $CY$.




Then in the triangles $PSY$, $WPY$, we have the sides $SP$, $PW$ equal by construction, $PY$ common, and the angles $SPY$,

angle; also the side $PY$ common: hence the triangles are equal in all respects, and $SP = WP$, and $SY = WY$, (Euc. I. 26.)

$$\therefore HW = HP - WP = HP - SP = 2AC \text{ (Prop. I.)}$$

Again in the triangles $SQY$, $WQY$, we have $SY = WY$, the side $QY$ common, and the included angles $SYQ$, $WYQ$ equal, each being a right angle; hence the triangles are equal in all respects, and $SQ = WQ$, (Euc. I. 4.)

$$\therefore HQ - WQ = HQ - SQ = 2AC \text{ (Prop. I.)}$$

$$\text{But } HW = 2AC, \therefore HQ - WQ = HW,$$

$$\text{or } HQ = WQ + HW,$$

which is impossible (Euc. I. 20).

Hence $PT$, or $PT$ produced, does not cut the hyperbola in any other point; therefore it is a tangent.

*WPY* equal by the property of the tangent; therefore the triangles are equal in all respects, and $\angle SYP = \angle WYP$, each of which is therefore a right angle. Hence $SY$ is the perpendicular on the tangent.

Again, $SC = CH$ and $SY = YW$, therefore $CY$ is parallel to $HW$; and by similar triangles $CSY$, $HSW$, $CY = \frac{1}{2}HW$.

But $HW = HP - PW = HP - SP = 2AC$.

Therefore $CY = AC$, and therefore $Y$ is a point in a circle the centre of which is $C$ and radius $AC$.

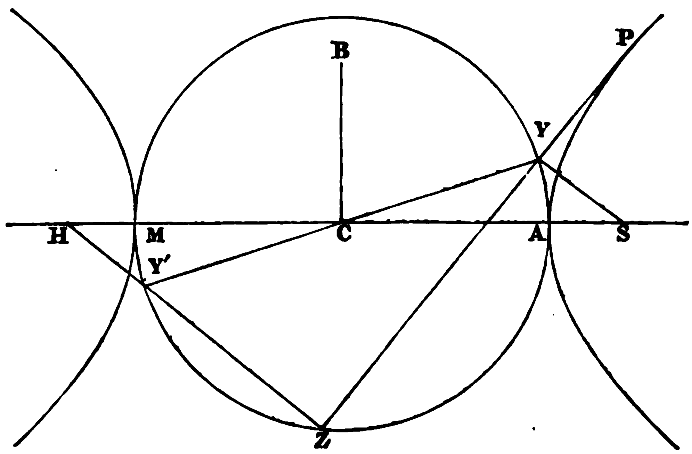
The proof would have been the same if we had considered $HZ$ the perpendicular from $H$, or if we had drawn the tangent to the other branch of the curve.

Cor. Draw the semiconjugate $CD$, cutting $HP$ in $E$: then $CYPE$ is a parallelogram, and $PE = CY = AC$.

### Prop. IV.

*The rectangle under the perpendiculars from the foci on the tangent is equal to the square of the semiaxis minor.*

$$(SY \,.\, HZ = BC^2.)$$




Let the perpendicular $HZ$ meet the auxiliary circle also in $Y'$. Join $CY$, $CY'$; then, because $YZY'$ is a right angle, therefore $YCY'$ is a diameter of the auxiliary circle, and $CY$,

$CY'$ are in the same straight line. Hence in the triangles $SCY$, $HCY'$, the sides $SC$, $CY$ are respectively equal to the sides $HC$, $CY'$ and $\angle SCY = \angle HCY'$: therefore the triangles are equal in all respects, and $SY = HY'$,

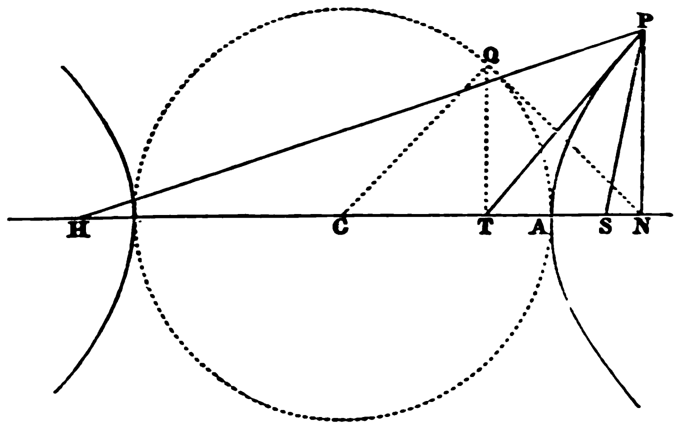
$$\therefore SY . HZ = HY' . HZ = HM . HA, \text{ (Euc. III. 36, Cor.)}$$
$$= HC^2 - CM^2. \text{ (Euc. II. 6.)}$$
$$= BC^2 \text{ by definition.}$$

Cor. By similar triangles $SYP$, $HZP$, (fig. Prop. III.)

$$SY : HZ = SP : HP,$$
$$\therefore SY^2 : SY . HZ = SP : HP,$$
$$\therefore SY^2 : BC^2 = SP : 2AC + SP.$$

## Prop. V.

*The rectangle under the lines intercepted between the centre and the intersections of the axis with the ordinate and tangent respectively is equal to the square of the semiaxis major.* ($CN . CT = AC^2$.)




Because $PT$ bisects the angle $HPS$,

$$\therefore HT : ST :: HP : SP, \text{ (Euc. VI. 3)};$$
$$\therefore HT - ST : HT + ST :: HP - SP : HP + SP,$$
$$\text{or } 2CT : SH :: 2AC : HP + SP.$$

Again, we have

$$SP^2 = SN^2 + PN^2,$$
$$HP^2 = HN^2 + PN^2;$$
$$\therefore HP^2 - SP^2 = HN^2 - SN^2,$$
$$\text{or } (HP + SP)(HP - SP) = (HN + SN)(HN - SN),$$
(Euc. II. 6),
$$\text{or } (HP + SP)\,2AC = 2CN \,.\, SH,$$
$$\text{or } 2CN : HP + SP :: 2AC : SH.$$

Hence $$CT : AC :: AC : CN,$$
$$\text{or } CT \,.\, CN = AC^2.$$

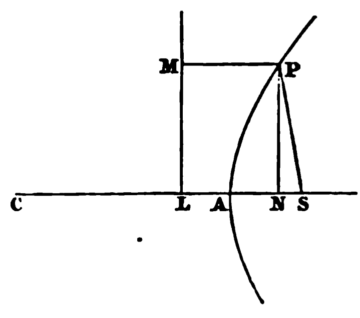
Cor. 1. Draw $TQ$ an ordinate to the auxiliary circle, and join $NQ$, $CQ$.

Then $$CT \,.\, CN = AC^2 = CQ^2,$$
$$\text{or } CT : CQ :: CQ : CN.$$

Therefore $CQN$ is a right angle, and $NQ$ *touches* the circle.

Cor. 2. From $P$ draw $PM$ perpendicular to the directrix $LM$. Then, as in the proposition,

$$(HP + SP)\,2AC = 2CN \,.\, SH,$$
$$\text{or } (2AC + 2SP)\,2AC = 2CN \,.\, 2CS,$$
$$\text{or } SP \,.\, AC = CN \,.\, CS - AC^2$$




But from the definition of the directrix and the proposition,

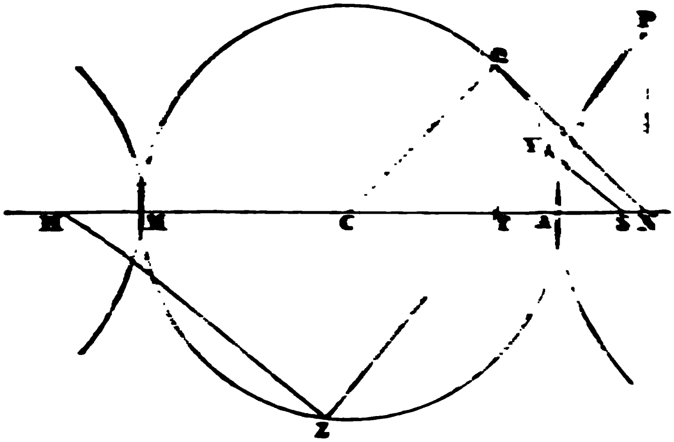
$$AC^2 = CS \,.\, CL = CS \,.\, CN - CS \,.\, NL;$$
$$\therefore SP \,.\, AC = CS \,.\, NL = CS \,.\, PM,$$
$$\text{or } SP : PM :: CS : AC.$$

In other words, the ratio of $SP$ to $PM$ is the same whatever may be the position of $P$ in the hyperbola.

## PROP. VI.

*The rectangle under the abscissae of the axis major is to the square of the semiordinate, as the square of the axis major to the square of the axis minor.*

$$(AN . NM : PN^2 = AC^2 : BC^2.)$$




Draw the tangent $PT$ and the perpendiculars upon it from the foci $SY$, $HZ$; draw $TQ$ an ordinate to the auxiliary circle, and join $NQ$, $CQ$. Then the triangles $SYT$, $PNT$, $HZT$ are similar;

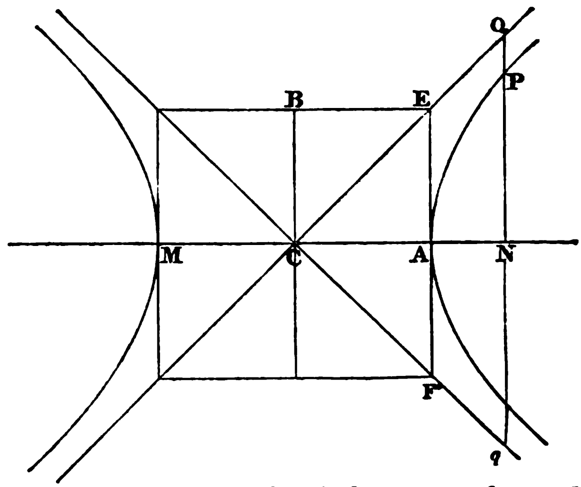
$$\therefore PN : SY :: NT : YT,$$
$$\text{and } PN : HZ :: NT : ZT;$$
$$\therefore PN^2 : SY . HZ :: NT^2 : YT . ZT,$$
$$\text{or } PN^2 : BC^2 :: NT^2 : QT^2, \text{ (Euc. III. 35)},$$
$$:: QN^2 : CQ^2, \text{ by similar triangles},$$
$$:: AN . NM : AC^2, \text{ (Euc. III. 36)},$$
$$\text{or } AN . NM : PN^2 :: AC^2 : BC^2.$$

COR. As in the case of the ellipse,

$$\cdot L . AC = 2BC^2.$$

## Prop. VII.

*If tangents be drawn at the vertices of the hyperbola and of the conjugate hyperbola, the diagonals of the rectangle so formed will be asympotes to the hyperbola.*




Let $CE$, $CF$ be diagonals of the rectangle, and let the ordinate $PN$ produced meet them in $Q$ and $q$ respectively.

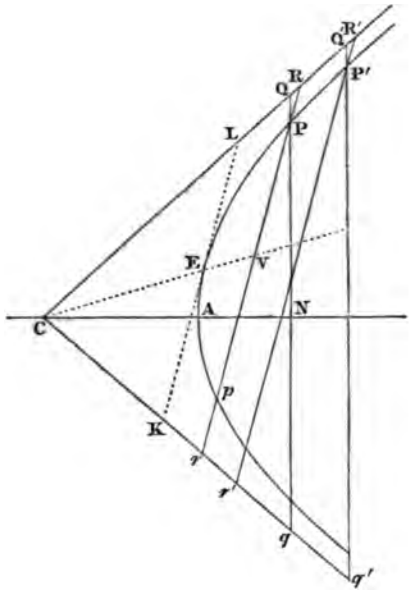
Then
$$NQ^2 : CN^2 :: AE^2 (BC^2) : AC^2,$$
$$:: PN^2 : AN.NM,$$
$$:: PN^2 : CN^2 - AC^2.$$
$$\therefore NQ^2 : PN^2 :: CN^2 : CN^2 - AC^2,$$
$$\text{or } NQ^2 - PN^2 : PN^2 :: AC^2 : CN^2 - AC^2,$$
$$\therefore QP.Pq : AC^2 :: PN^2 : CN^2 - AC^2,$$
$$:: BC^2 : AC^2,$$
$$\text{or } QP.Pq = BC^2.$$

Now the rectangle $QP.Pq$ being always equal to $BC^2$, and the side $Pq$ continually increasing, the side $QP$ must continually diminish; but however far from $A$ we take the point $P$, $QP$ can never be actually zero; hence the straight line $CE$ produced continually approaches the hyperbola and the distance between them becomes less than any assignable quantity, but it never actually meets the curve; that is, it is an asymptote.

**Cor.** The same line is an asymptote to the conjugate hyperbola.

## Prop. VIII.

***If any straight line be drawn making a given angle with the axis, and terminated by the asymptotes, the rectangle under the segments into which it is divided by the curve is invariable.***




If the straight line be perpendicular to the axis we may prove, as in the preceding proposition, that

$$QP \cdot Pq = BC^2.$$

But if it be not perpendicular to the axis, as $RPr$, then take any other point $P'$ in the hyperbola, and through it draw $Q'P'q'$ parallel to $QPq$, and $R'P'r'$ parallel to $RPr$.

Then by similar triangles,

$$RP : R'P' :: QP : Q'P',$$
and $$Pr : P'r' :: Pq : P'q',$$

$$\therefore RP \cdot Pr : R'P' \cdot P'r' :: QP \cdot Pq : Q'P' \cdot P'q',$$
$$:: BC^2 : BC^2.$$
$$\therefore RP \cdot Pr = R'P' \cdot P'r'.$$

**Cor. 1.** If the line $RPr$ cuts the curve also in $p$, we have, manifestly,

$$RP \cdot Pr = Rp \cdot pr,$$
or $$RP \cdot Pp + RP \cdot pr = RP \cdot pr + Pp \cdot pr;$$
$$\therefore RP = pr.$$

**Cor. 2.** The same conclusions will hold, if we suppose $RPr$ to move parallel to itself until it becomes a tangent $LEK$. Hence we have $LE = EK$, and $RP \cdot Pr = LE^2$.

COR. 3. Join $CE$, and let it when produced cut $Pp$ in $V$. Then,

$$VR : LE :: VC : EC$$
$$:: Vr : EK,$$

but $LE = EK$, $\therefore VR = Vr$; also $RP = rp$; therefore $PV = Vp$, or a diameter bisects its ordinates.

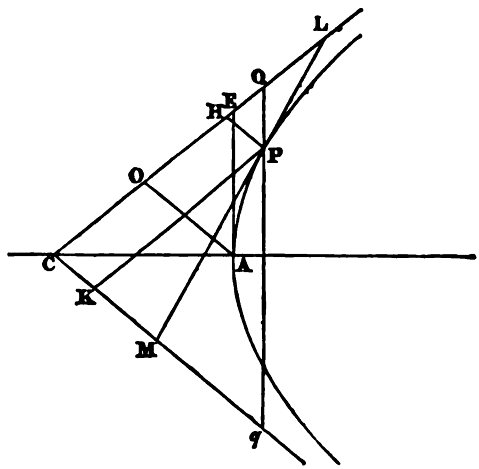
## PROP. IX.

*If from any point in the curve straight lines are drawn parallel to and terminated by the asympotes, their rectangle is invariable.*

Let $PH$, $PK$ be the lines parallel to the asympotes. Draw the tangent $LPM$, and $QPq$ perpendicular to the axis, $AE$ the tangent at the vertex, and $AO$ parallel to the asymptote.

Then, by similar triangles,

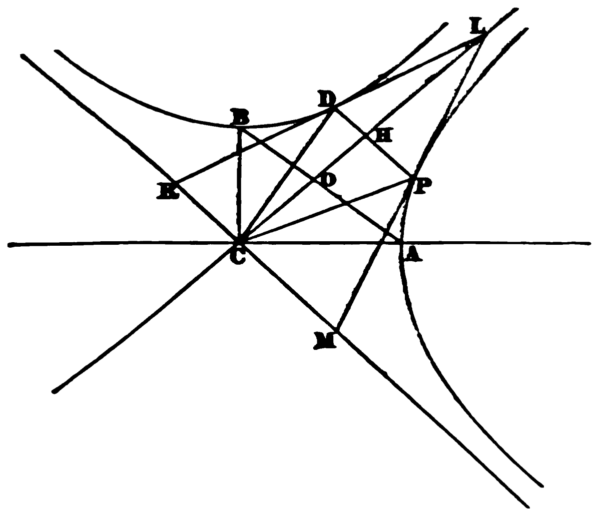
$$PH : PQ :: AO : AE,$$
$$PK : Pq :: OE : AE;$$




$$\therefore PH.PK : PQ.Pq\,(BC^2) :: AO.OE\,(AO^2) : AE^2\,(BC^2),$$
$$\text{or } PH.PK = AO^2$$
$$= \tfrac{1}{4}(AC^2 + BC^2).$$

COR. 1. The parallelogram $PHCK$ is constant; and therefore also the triangle $LCM$ is constant, for since $PL = PM$ it is double of the parallelogram $PHCK$.

COR. 2. A straight line drawn parallel to one asymptote and terminated by the conjugate hyperbolas, is bisected by the other asymptote.




Let $PHD$ be such a line, then by the proposition

$$PH.HC = \tfrac{1}{4}(AC^2 + BC^2),$$

similarly for the conjugate hyperbola,

$$DH.HC = \tfrac{1}{4}(AC^2 + BC^2);$$

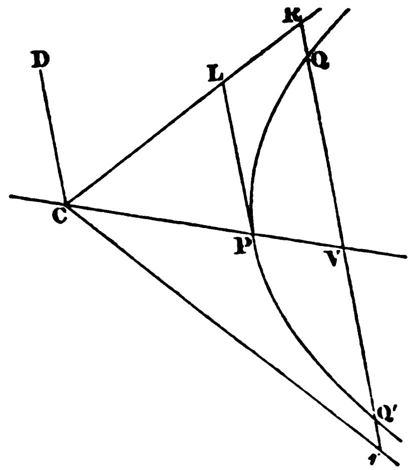
$$\therefore PH = DH.$$

COR. 3. If we draw the tangent $LPM$, it is bisected in $P$, and therefore $CL = 2CH$; hence the tangent $RDL$ to the conjugate hyperbola at $D$, must meet the asymptote in the same point $L$. Also $CM = 2HP = DP$, therefore $CDPM$ is a parallelogram, and $CD$ is the semi-diameter conjugate to $CP$.

COR. 4. $RL = 2CP =$ the diameter at $P$; and $LM = 2CD$ = the diameter conjugate to $CP$.

## Prop. X.

*The rectangle under the abscissæ of any diameter is to the square of the semi-ordinate, as the square of the diameter to the square of the conjugate.* ($PV.VG : QV^2 :: CP^2 : CD^2$.)




Let $QVQ'$ produced meet the asymptotes in $R$, $r$; and draw $PL$ a tangent at $P$ terminated by the asymptote, then $PL$ equals the semi-conjugate $CD$.

Now

$$RQ.Qr = RV^2 - QV^2 = PL^2,$$

$$\therefore QV^2 = RV^2 - PL^2;$$

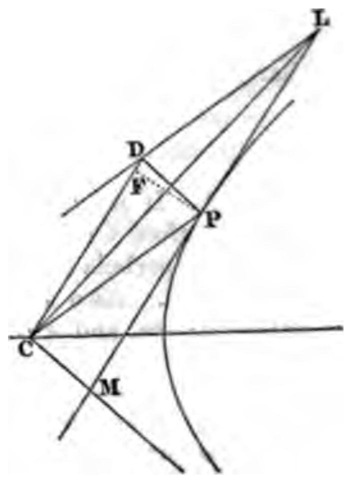
but $CV^2 : CP^2 :: RV^2 : PL^2$;

$$\therefore CV^2 - CP^2 : RV^2 - PL^2 :: CP^2 : PL^2,$$

or $PV.VG : QV^2 :: CP^2 : CD^2$.

## Prop. XI.

*The parallelograms formed by tangents at the vertices of pairs of conjugate diameters have all the same area.*




Let $MPL$, $DL$ be the tangents, $CL$, $CM$ the asymptotes; then the parallelogram in this case

$$= 4\,CDLP = 8\,CLP = 4LCM,$$

which is constant, since $LCM$ is constant. (Prop. IX. Cor. 1.)

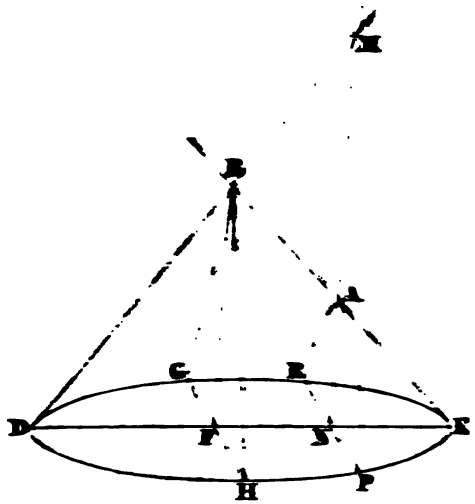
Cor. Draw $PF$ perpendicular to $CD$, then

$$PF.CD = CDLP;$$

but when the tangents are drawn at $A$ and $B$, the area of the parallelogram $CDLP$ becomes $AC.BC$, therefore $PF.CD = AC.BC$.

PROP. [illegible]

*If a right cone is cut by a plane* [illegible] *both sides of the vertex, the section is an hyperbola.*




Let *RAP* be the section, which is supposed to be perpendicular to the plane of the paper; *DGEH* a section of the cone perpendicular to the axis, which is therefore circular, *RNP* the intersection of the planes *RAP*, *DGEH*, which is manifestly perpendicular to both *DE* and *AN*, *BGH* a triangular section through the vertex of the cone by a plane parallel to *RAP*.

$$\text{Then} \quad AN : EN :: BF : EF,$$
$$NM : ND :: BF : FD,$$
$$\therefore AN \,.\, NM : EN \,.\, ND :: BF^2 : EF \,.\, FD,$$
$$\text{or } AN \,.\, NM : PN^2 :: BF^2 : FH^2,$$

which is the property of an hyperbola, the major axis of which is *AM*, and the minor axis is to *AM* as *FH* : *BF*; hence the section is an hyperbola.

NOTE. For some other propositions concerning the Conic Sections, see the Digression concerning the curvature of curves in the First Section of Newton's Principia.

## SCHOLIUM.

Although, throughout this treatise, the symbol $AB.CD$ has been used to express the rectangle under the two lines $AB$, $CD$, and not to express as in Algebra the multiplication of $AB$ by $CD$, nevertheless the symbol may be regarded in this algebraical point of view: for if $a$ be the number of units of length in $AB$, and $b$ the number in $CD$, then will $a \times b$ be the number of units of area in the rectangle $AB.CD$, and therefore $AB.CD$ may be regarded as a product; it being understood that in so regarding it, $AB$ represents the number of units of length in the line $AB$, and $CD$ the number in the line $CD$. In like manner the ratio $AB : CD$ may be written thus $\frac{AB}{CD}$, with the same understanding. The importance of this Scholium will be appreciated by the student, when he sees the application of the properties of the Conic Sections in the sequel.

# MECHANICS.

I. STATICS.

II. DYNAMICS.

# MECHANICS.

The science of *Mechanics* treats of the effects of *force;* we must therefore commence by explaining what we mean by *force.*

*Force is any cause which changes, or tends to change, a body's state of rest or motion.*

By the term *body* here used, we intend to express any material substance, or portion of matter; we cannot conceive of the action of a force except as taking place upon a material body; and the body may be of various kinds, but at present we shall concern ourselves only with the action of force upon a *particle,* and the action of force upon a *rigid body.* By a *particle* we intend, without entering into any discussion respecting the ultimate constitution of matter, to designate the smallest quantity of matter conceivable, so that we need not concern ourselves with its shape or its magnitude; nevertheless all particles are not necessarily equal, for though each of two particles be indefinitely small, one may be greater than the other in any proportion. By a *rigid body* we mean to denote any portion of matter, the constituent particles of which are so connected as to be incapable of changing their relative position. From this definition it is easy to see, that no such thing as a mathematically rigid body exists in nature, for the hardest known substances are susceptible of compression and extension under the action of great pressures; steel, for instance, though a very hard substance, is not rigid; nevertheless the conclusions which we shall arrive at in the following treatise, though mathematically true only of rigid bodies, will be practically true of all ordinary solid bodies: for, to consider a body as absolutely rigid, or incapable of changing its form under the action of any force, is the same thing practically as to suppose, that no forces are called into

play, which actually do produce any sensible change of form in the solid body under consideration.

When any number of forces act upon a material body they will produce one of two effects, they will either keep the body at rest or they will cause motion, and these two effects are of such very distinct kinds that they require to be treated separately; and thus the science of Mechanics naturally divides itself into two parts, the first and more simple of which treats of forces which keep a body at *rest*, or are in *equilibrium*, and is called *Statics;* the second, of forces which produce *motion*, and is called *Dynamics*.

# STATICS.

1. **Force** is measured statically by the pressure which will counteract it. So far as the principle of measuring force is concerned, it is indifferent what kind of pressure we choose as the standard by which to measure force; there is one kind of pressure, however, which in the nature of things is more convenient than any other, and that is *weight*. We may conceive then of a force as being measured by the number of pounds which it can lift, and we may compare two forces by means of the numbers of pounds which they can lift respectively.

In what follows we shall denote the *magnitude* or *intensity* of force by letters, such as $P$, or $Q$, or $R$; when we speak of *a force* $P$, all that is intended is that the force in question would just lift $P$ lbs., and therefore has $P$ lbs., or more shortly $P$, for its statical measure.

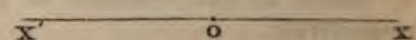
2. In order that we may be able to calculate the combined effect of any number of forces acting on a particle, it is not sufficient that we should know the intensity of each; for it is obvious that the effect of a force depends upon the *direction* in which it acts as well as its magnitude, and that a system of forces cannot in general be in equilibrium unless their directions as well as their magnitudes satisfy certain conditions, which conditions the science of Statics must teach us. Let us endeavour to form a distinct conception of the *direction* of a force: suppose a force to act upon a particle at rest, and the particle to be prevented from moving by a string, one end of which is fixed and the other attached to the particle, then the direction of the force coincides with the string; or we may say, that the direction of a force is that, in which the particle would begin to move, if not constrained to remain at rest. The direction of a force is sometimes called the *line of its action*.

3. The *intensity* and *direction* are the only elements necessary to entirely describe any force which acts on a *particle*, but if we consider its action on a rigid body we shall require in addition to know the *point of its application:* at present we shall concern ourselves only with the action on a single particle.

The simplest case of such action is when there are two forces only, and it is manifest that, in this case, there can be equilibrium only when the two forces are equal in intensity and exactly opposite in their direction: that is, if two forces $P$ and $Q$ are in equilibrium, they must act in the same line and tend to draw the particle opposite ways, and we must have the condition,

$$P = Q, \text{ or } P - Q = 0.$$

4. We may here observe, that the method, which was adopted in the Treatise on Trigonometry, of denoting *direction* by an algebraical sign, may be applied to forces; that is, if we denote by $+P$ a force acting on a particle at $O$ in the direction $OX$, then a force of the same magnitude, but acting in the opposite direction $OX'$, will be properly denoted by $-P$. Hence we may say, that if any number of forces act along the same straight line on a particle, the condition of equilibrium is, that their *algebraical* sum shall be zero; and if the sum be not zero, then the force represented by it will be the *resultant* of the forces acting on the particle, and will tend to draw the particle in the direction $OX$, or in the direction $OX'$, according as its sign is + or −.




For instance, suppose we have two forces $P$, $Q$ acting on a particle at $O$ in the direction $OX$, and two others $R$, $S$ in the direction $OX'$, then the resultant will be $P + Q - R - S$, the algebraical sum of the four forces; and in order that there may be equilibrium, we must have,

$$P + Q - R - S = 0.$$

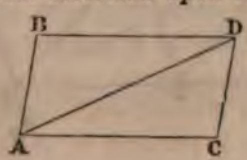
Or, for simplicity's sake, to take a numerical example. Suppose we have two forces, represented by 2 and 3 lbs., respectively, acting on a particle at $O$ in the direction $OX$,

and a force of 4 lbs. acting in the direction $OX'$, then the *resultant* will be the force $(2 + 3 - 4 =)$ 1 lb. in the direction $OX$; or if we have a force of 1 lb. acting in the direction $OX'$ there will be equilibrium.

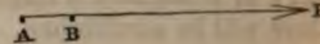
5. When two forces act on a particle, not along the same straight line, they cannot be in equilibrium, but will be equivalent to some one force which we shall call their *resultant:* that they are equivalent to some one force is manifest from the consideration, that a particle under the action of two forces would *begin* to move in a certain definite direction, and that it may be prevented from moving by a string attached to it and coinciding with that direction; the string spoken of will undergo a certain *tension*, and that tension or the weight which would produce it measures the magnitude of the resultant of the two forces.

6. The fundamental problem of Statics, is to find the magnitude and direction of the resultant of two forces; but in order to solve it, we must premise that it is convenient to represent *forces* by straight *lines;* and it is manifest that we may by means of a line represent a force, as to both magnitude and direction; for we can represent the magnitude, by taking a line which bears the same proportion to some standard length, (as for instance 1 inch,) as the force bears to the standard pressure, (or 1 lb.); and the direction will be represented, by drawing the line in that direction in which the force tends to make the particle move. By means of this convenient mode of representing forces, we are able to enunciate the relation between two forces and their resultant in the form of the following Theorem, which is known as the PARALLELOGRAM OF FORCES.

*If two forces acting on a particle at* A, *be represented in direction and magnitude by the lines* AB, AC, *then the resultant will be represented in direction and magnitude by the diagonal* AD *of the parallelogram described upon* AB, AC.



7. The proof which we shall give of this proposition depends upon this principle; *A force may be supposed to act at any point in its direction, that point being conceived to be rigidly attached to the particle on which the force acts.* Thus, if we have a force $P$ acting on a particle at $A$, we may suppose, if we please, that the force acts at $B$, $B$ being rigidly connected with $A$; this is a principle which the student will have no difficulty in grasping, and which may be illustrated roughly by saying, that the force required to toll a bell is independent of the length of the rope, and the effort required to move a carriage independent of the length of the traces.




We are now able to give the following

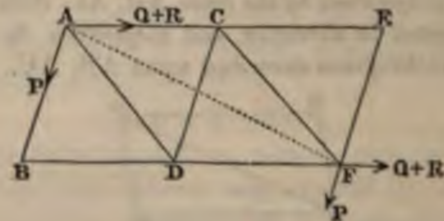
PROOF OF THE PARALLELOGRAM OF FORCES.

8. I. To prove the proposition so far as the *direction* of the resultant is concerned.

When the forces are *equal*, it is manifest that the direction of the resultant will bisect the angle between the directions of the forces: or, if we represent the forces in direction and magnitude by two lines drawn from the point at which they act, the diagonal of the parallelogram described upon these lines will be the direction of the resultant.

Next, suppose that the proposition, just proved for equal forces, is true for two unequal forces $P$ and $Q$, and also for $P$ and $R$: we shall shew that it will be true for $P$ and $Q + R$.

Let $A$ be the point of application of the forces; take $AB$ to represent $P$ in direction and magnitude, and $AC$ to represent $Q$; complete the parallelogram $ABDC$, then by



hypothesis $AD$ is the direction of the resultant of $P$ and $Q$; and, since a force may be supposed to act at any point of its direction, we may consider $D$ as the point of application of the resultant of $P$ and $Q$; or we may suppose the forces $P$ and $Q$ themselves to act at $D$, $P$ parallel to $AB$ and $Q$ to $AC$; or still further, we may suppose $P$ to act at $C$ in the direction $CD$.

Again, the force $R$ which acts at $A$ may be supposed to act at $C$; take $CE$ to represent it in direction and magnitude, and complete the parallelogram $CDFE$, then by hypothesis $CF$ is the direction of the resultant of $P$ and $R$ which acted at $C$: hence the resultant of $P$ and $R$ may be supposed to act at $F$, or $P$ and $R$ may themselves be supposed to act at that point, parallel to their original directions.

Lastly, the force $Q$, which is at present supposed to be acting at $D$ in the direction $DF$, may be supposed to act at $F$.

Hence we have reduced the forces $P$ and $Q + R$ acting at $A$, to $P$ and $Q + R$ acting in the same directions at $F$; consequently $F$ is a point in the line of action of the resultant, and therefore $AF$ is the direction of the resultant: that is, if the proposition be true for $P$ and $Q$, and also for $P$ and $R$, it is true for $P$ and $Q + R$.

But the proposition is true for $P$ and $P$, and also for $P$ and $P$, therefore it is true for $P$ and $2P$, therefore for $P$ and $3P$, and so on; therefore generally for $P$ and $mP$.

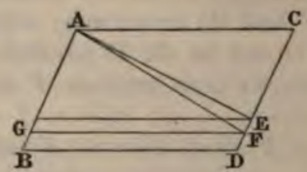
In like manner the proposition may be extended to $mP$ and $nP$, ($m$ and $n$ being whole numbers,) that is, to any *commensurable forces**.

The proposition is extended to incommensurable forces as follows.

Let $AB$, $AC$ represent any two incommensurable forces; complete the parallelogram $ABDC$, and if $AD$ is not the direction of the resultant, let it be $AE$. Suppose $AC$ to be divided into a number of equal parts, each part being less than $ED$, and suppose distances of the same magnitude to

* Two quantities are said to be *commensurable* when their ratio can be expressed by the ratio of two *whole numbers*.

be set off along $CD$ beginning at $C$, then one of the divisions must fall between $E$ and $D$; let $F$ be the point which marks




the division, and complete the parallelogram $AGFC$, then $AF$ is the direction of the resultant of the *commensurable* forces $AG$, $AC$: but $AF$ makes a larger angle with $AC$ than $AE$, that is, the resultant of $AG$ and $AC$ lies further away from $AC$ than the resultant of $AB$ and $AC$, although $AG$ is less than $AB$, which is absurd: hence $AE$ is not the direction of the resultant, and it may be shewn in like manner that no other line is in that direction except $AD$. Hence the proposition, which was proved for *commensurable* forces, is true for *incommensurable*.

II. To prove the parallelogram of forces with respect to the *magnitude* of the resultant.

Let $AB$, $AC$ represent the forces; complete the parallelogram $ABDC$, join $DA$ and produce it to $D'$, making $AD'$ *equal* to the resultant of $AB$ and $AC$ *in magnitude*; complete the parallelogram $ABC'D'$, and join $AC'$.


Then since $AD'$ is equal to the resultant of $AB$ and $AC$, and drawn in the direction opposite to that of their resultant, the three forces $AB$, $AC$, $AD'$ will balance each other, and therefore any one of them is in the direction of the resultant of the other two; hence $AC$ is in the direction of the resultant of $AB$, $AD'$; but $AC'$ is also in that direction, therefore $AC$, $AC'$ are in the same straight line. Hence $ADBC'$ is a parallelogram; therefore $AD = BC'$: but $BC' = AD'$, therefore $AD = AD'$. And by

construction $AD'$ represents the resultant of $AB$ and $AC$ in magnitude, therefore $AD$ also represents the resultant, and the proposition enunciated is true.

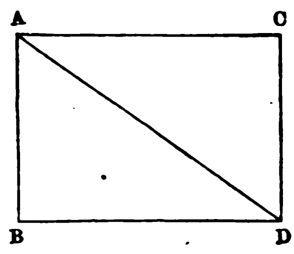
9. The proposition, which we have now established, enables us to reduce any system of forces acting on a particle to one single force; for we can find the resultant of any two of the forces, then of that resultant and a third, and so on.

10. As the parallelogram of forces enables us to compound two forces into one, so, conversely, we are able by means of it to *resolve* any force into two, that is, to find two forces which shall be equivalent to a given force. This is a problem which obviously admits of an infinite number of solutions; in fact, if upon a line representing the given force in direction and magnitude, as diagonal, we describe any parallelogram, the sides of that parallelogram will represent forces, which by their composition are equivalent to the given force.

Before proceeding further, we shall supply the student with a few examples of the composition and resolution of forces.

Ex. 1. Two forces, measured by 3lbs. and 4lbs. respectively, act on a particle, at right angles to each other; find the magnitude of their resultant.

If $AB$, $AC$ represent the forces, and we complete the rectangle $ABDC$, we have

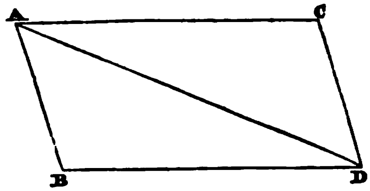



$$AD^2 = AB^2 + AC^2$$
$$= 3^2 + 4^2$$
$$= 9 + 16 = 25;$$
$$\therefore AD = 5,$$

or the measure of the resultant is 5 lbs.

Ex. 2. Two forces, 1 and 2 lbs. respectively, act at an angle of $60^\circ$; find the direction and magnitude of their resultant.

Let $AB$ and $AC$ represent the forces, $AD$ their resultant, and let $BAD = \theta$. Then by the data of the problem, $BAC = 60^0$.



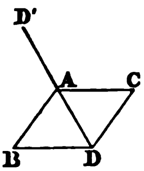

$$\therefore AD^2 = AC^2 + CD^2 - 2AC \,.\, CD \cos ACD$$
$$= AC^2 + AB^2 + 2AC \,.\, AB \cos BAC$$
$$= 4 + 1 + 4 \cos 60^0$$
$$= 5 + 2, \text{ (since } \cos 60^0 = \tfrac{1}{2})$$
$$= 7;$$
$$\therefore AD = \sqrt{7}.$$

Again, from the triangle $ABD$,

$$\frac{\sin\theta}{\sin ABD} = \frac{BD}{AD},$$

$$\text{or, } \sin\theta = \frac{2}{\sqrt{7}} \sin 60^0 = \sqrt{\tfrac{3}{7}}.$$

Ex. 3. Three forces $P$, $Q$, $R$ are in equilibrium; find the angles between their directions.




Let $AB$, $AC$ represent $P$ and $Q$ respectively; complete the parallelogram $ABDC$, and produce $DA$ to $D'$, making $AD' = AD$; then $AD'$ represents $R$.

Let $BAC = \theta$; $\therefore ACD = 180^0 - \theta$,

$$\text{and } AD^2 = AC^2 + CD^2 + 2AC \,.\, CD \cos\theta;$$

$$\therefore \cos\theta = \frac{AD^2 - AC^2 - CD^2}{2AC \,.\, CD}$$
$$= \frac{R^2 - P^2 - Q^2}{2PQ},$$

which determines the angle between the directions of $P$ and $Q$. The angles between the directions of $P$ and $R$, $Q$ and $R$, are known in like manner.

Ex. 4. As a further illustration we will shew how the direction of the normal or tangent to an ellipse may be deduced from the consideration of the equilibrium of three forces acting at one point.

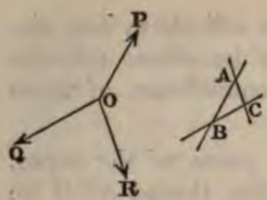
Let $S$, $H$ be two fixed points in the plane of the paper, and to them let the extremities of a fine thread $SPH$ be fastened, upon which a small ring or bead $P$ runs freely. Then if $P$ be made to move in such a manner as to keep the portions of thread $SP$, $HP$ stretched, its locus will be an ellipse.

Now suppose we have a force $R$, the tension of another thread for instance, acting upon the ring in such a manner as to keep it at rest. Then we shall have the ring at rest under the action of the force $R$ and the tensions of the two portions of thread $SP$ and $HP$: but these tensions must be equal, since the tension of the thread cannot be different at different points. Hence the direction of $R$ must make equal angles with the two portions of the thread.

But it is impossible that the ring should be at rest, unless the force $R$ be perpendicular to the direction in which the ring would move, if it moved at all: that is, the direction of $R$ must be normal to the ellipse which is the locus of $P$.

Therefore the normal to the ellipse which is the locus of $P$ bisects the angle between the focal distances $SP$, $HP$.

11. The parallelogram of forces may be stated in another form, under which it is called the *Triangle of Forces.* For we have seen that three forces will be in equilibrium, provided they are proportional to the sides and diagonal of a parallelogram, and act on a particle parallel to those sides; but the sides and diagonal form a triangle; indeed, it is the same thing whether we say of three straight lines that they are the sides and diagonal of a parallelogram, or that they form a triangle; hence we may assert, that forces will be in equilibrium, when they are proportional to the sides of a triangle formed by drawing lines parallel to their directions.

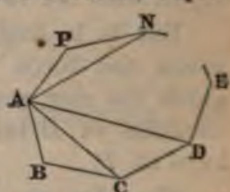



Suppose, in fact, that the forces $P$, $Q$, $R$ are in equilibrium on the particle $O$; draw any three straight lines parallel to the directions of the three forces, and let $ABC$ be the triangle formed by their intersections, then

$$P : Q : R :: AB : BC : CA.$$

Hence it may be said that if two sides ot a triangle taken in order from an angular point represent in magnitude and direction two forces which act at that point, then the third side, not taken in the same order as the other two, will represent the resultant. Thus if $AB$, $BC$ represent two forces acting at $A$, then $AC$, (not $CA$,) will represent the resultant.

And this proposition may be generalised so as to assume a form under which it may be called the *Polygon of Forces:* thus, if any number of lines $AB$, $BC$, $CD$, $DE \ldots NP$, represent in magnitude and direction forces acting at $A$, then the line $AP$ completing the polygon will represent the resultant. It follows, of course, that if a particle be acted upon by forces which can be represented by the sides of a polygon taken in order, the particle will be at rest. It may be observed that the lines forming the polygon need not lie in one plane.




Cor. If $\alpha$, $\beta$, $\gamma$ are the angles between the directions of $Q$ and $R$, $R$ and $P$, $P$ and $Q$ respectively, then

$$P : Q : R :: \sin\alpha : \sin\beta : \sin\gamma;$$

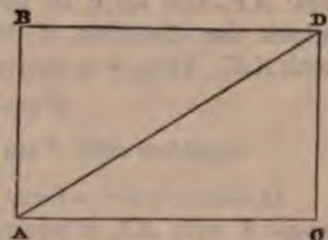
or, (as it may be otherwise written,)

$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}.$$

12. It follows from the triangle of forces, that any conclusions established concerning the relations of the sides and

angles of a triangle, may be extended to the magnitudes and directions of forces in equilibrium. For instance, we may conclude from Euclid, I. 20, that of three forces in equilibrium any two must be greater than the third.

13. We have seen that a force in a given direction may always be replaced by two forces in two other directions, which forces are called the *components* of the original force. There is a peculiarity in the case of these components being at right angles to each other, which requires notice. Let $AD$ represent a force, which is resolved into the two rectangular components $AB$, $AC$. Then it is manifest, that the force $AB$ has no tendency to move the particle in the direction $AC$, neither has $AC$ any tendency to move it in the direction $AB$. Hence we may say that $AB$, $AC$ measure the *whole* effect of $AD$ in the directions $AB$, $AC$ respectively, and they are usually termed the *resolved parts* of $AD$.




If we call the angle $BAD$ $\theta$, we have

$$AB = AD \cos\theta,$$

$$AC = AD \sin\theta.$$

Hence, if $X$ be the *resolved part* of a force $P$ in a direction making an angle $\theta$ with the direction of $P$, and $Y$ the resolved part in the direction perpendicular to that of $X$, we shall have

$$X = P\cos\theta,$$

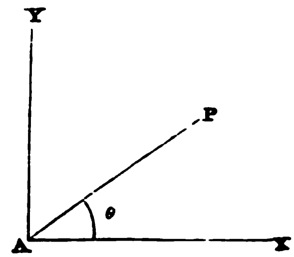
$$Y = P\sin\theta.$$

Also, $\tan\theta = \frac{Y}{X}$, and $X^2 + Y^2 = P^2$.

The preceding formulæ may be looked upon as fundamental in Statics; they enable us to solve the following most general problem.

14. *Any number of forces act at the same point, their directions all lying in the same plane; find the direction and magnitude of their resultant.*

Let $P$ be any one of the forces acting at the point $A$. Let the plane of the paper be that in which the forces act; in that plane choose any two lines at right angles to each other, $AX$ and $AY$, and let $\theta$ be the angle which the direction of $P$ makes with $AX$. Then $P$ is equivalent to




$P \cos\theta$ acting in the direction $AX$,

together with $P \sin\theta$ ........................ $AY$.

In like manner, a force $P'$, the direction of which makes an angle $\theta'$ with $AX$, is equivalent to

$P' \cos\theta'$ acting in the direction $AX$,

together with $P' \sin\theta'$ ........................ $AY$.

And so on for any number of forces. Hence, adding together the forces which act in the same direction, we shall have a system of forces $P, P' \ldots$ acting at angles $\theta, \theta' \ldots$ with the line $AX$, equivalent to

$P \cos\theta + P' \cos\theta' + \ldots$ acting in the direction $AX$,

together with $P \sin\theta + P' \sin\theta' +$ ........................ $AY$.

For shortness' sake, let

$$P \cos\theta + P' \cos\theta' + \ldots\ldots = X,$$

$$P \sin\theta + P' \sin\theta' + \ldots\ldots = Y,$$

and let $R$ be the required resultant, $\phi$ the angle which its direction makes with the line $AX$; then

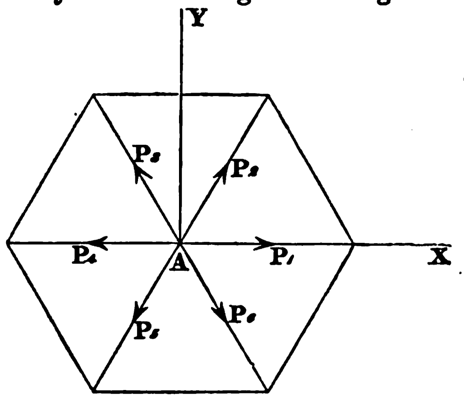
$$R \cos\phi = X,$$

$$R \sin\phi = Y;$$

$$\therefore \tan\phi = \frac{Y}{X}, \; R^2 = X^2 + Y^2.$$

These formulæ determine the direction and magnitude of the resultant of the system of forces.

Ex. A particle is placed at the centre of a regular hexagon, and is acted upon by forces tending to the angles of the hexagon and measured by $P_1$, $P_2$, $P_3$, $P_4$, $P_5$, $P_6$ respectively; determine the direction and magnitude of the resultant force.




If we choose the line $AX$ so as to pass through one of the angular points, it will be easily seen that we shall have in this example, $\left(\text{observing that } \cos 60^\circ = \frac{1}{2}, \text{ and } \sin 60^\circ = \frac{\sqrt{3}}{2}\right)$,

$$X = P_1 + P_2 \tfrac{1}{2} - P_3 \tfrac{1}{2} - P_4 - P_5 \tfrac{1}{2} + P_6 \tfrac{1}{2}$$
$$= P_1 - P_4 + \tfrac{1}{2}(P_2 - P_3 - P_5 + P_6),$$
$$Y = \left(\frac{P_2}{2} + \frac{P_3}{2} - \frac{P_5}{2} - \frac{P_6}{2}\right)\sqrt{3}.$$

Suppose, for instance, $P_1 = 1$, $P_2 = 2$, $P_3 = 3$, $P_4 = 4$, $P_5 = 5$, $P_6 = 6$, then

$$X = 1 - 4 + \tfrac{1}{2}(2 - 3 - 5 + 6) = -3,$$
$$Y = \left(\frac{2 + 3 - 5 - 6}{2}\right)\sqrt{3} = -3\sqrt{3};$$
$$\therefore \tan\phi = \sqrt{3}, \text{ or } \phi = 60^\circ;$$
$$R^2 = 9 + 27 = 36, \text{ or } R = -6.$$

Thus the resultant is completely determined; we take the negative value of $R$, because since $X = R \cos\phi$, and in this case $X$ is negative and $\cos\phi$ positive, $R$ must be negative. It would have come to the same thing if we had taken $\phi = 240^\circ$, and $R$ positive.

15. *To find the conditions of equilibrium of any system of forces, acting in one plane at the same point.*

Suppose the forces are reduced to one ($R$), as in the last article; then in order that there may be equilibrium we must have

$$R = 0,$$

$$\text{or } X^2 + Y^2 = 0.$$

And this equation is equivalent to these two,

$$X = 0, \qquad Y = 0,$$

$$\text{or } P\cos\theta + P'\cos\theta' + \ldots\ldots = 0,$$

$$P\sin\theta + P'\sin\theta' + \ldots\ldots = 0.$$

These are the conditions of equilibrium, which may be expressed in words by saying, that *the sum of the forces resolved in any two directions perpendicular to each other must vanish.*

## ON THE PRINCIPLE OF THE LEVER.

16. Hitherto we have considered forces acting on a particle only; when we come to the consideration of the action of forces on a rigid body, there will be other conditions of equilibrium besides those already deduced. In the case of a single particle the only necessary condition is that there shall be no motion of *translation;* but in order that a rigid body may be at rest it is not sufficient that any one point in it should be fixed, it is also necessary that there should be no *twisting* about that point. The simplest case is that of two forces acting on a rigid rod, one point of which is fixed; supposing that the forces tend to twist the rod in opposite ways, we can find the conditions under which they will counteract each the effect of the other, and produce equilibrium.

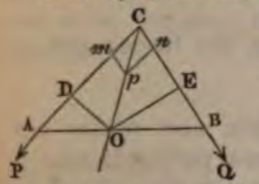
Def. A rigid rod, moveable about a fixed point in its length, is called a *lever.*

Def. The fixed point is called the *fulcrum,* and the distances between the fulcrum and the extremities the *arms.*

DEF. The *moment* of a force with respect to a given point, is the product of the force and the perpendicular from the point on its direction.

17. PROP. *If two forces acting at the extremities of a lever and tending to twist the lever opposite ways produce equilibrium, the moments of the forces about the fulcrum are equal.*

Let $P$, $Q$ be the two forces acting at $A$ and $B$, the extremities of a lever. Produce the directions of $P$ and $Q$ until they meet in $C$, then $P$ and $Q$ may both be supposed to act at $C$: take $Cm$, $Cn$ proportional to $P$ and $Q$, and complete the parallelogram $Cmpn$; join $Cp$ and produce it to cut $AB$ in $O$, then the resultant of $P$ and $Q$ acts in the direction $CO$, and therefore $O$ must be the fulcrum, otherwise there could not be equilibrium. Draw $OD$, $OE$ perpendicular to $AC$, $BC$; then



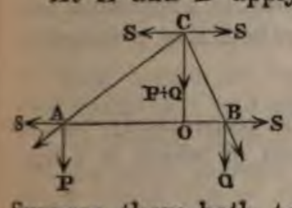

$$\frac{P}{Q} = \frac{\sin Cpm}{\sin mCp} = \frac{CO \sin BCO}{CO \sin ACO} = \frac{OE}{OD};$$

$$\therefore P \,.\, OD = Q \,.\, OE,$$

or the moments of $P$ and $Q$ about $O$ are equal.

If the forces act parallel to each other, their directions will not meet as they are supposed to do in the preceding proposition; in this case we must proceed as follows. We shall suppose, though it is not necessary, that the forces act perpendicular to the lever.

At $A$ and $B$ apply any two equal and opposite forces $S$ in the direction of the lever; this will manifestly not affect the equilibrium; then the resultant of $P$ and $S$ will be some force in the direction $CA$ suppose, and that of $Q$ and $S$ some force in the direction $CB$. Suppose them both to act at $C$, and there to be resolved into their constituent parts $P$ and $S$, $Q$ and $S$; the portions $S$, $S$ will destroy each other, leaving a resultant $P + Q$ in the direction $CO$ parallel to the directions of the forces.



Then the sides of the triangle $AOC$ are parallel to the directions of the forces $P$, $S$ and their resultant;

$$\therefore \frac{P}{S} = \frac{CO}{AO},$$

in like manner, $$\frac{Q}{S} = \frac{CO}{BO};$$

$$\therefore P \,.\, AO = Q \,.\, BO.$$

The same method is applicable, when the forces are not perpendicular to the arm.

If we had first proved the proposition now under consideration for the case of parallel forces perpendicular to the arm, it would have been very easy to deduce the more general case. For if in the first figure of page 219 we suppose the forces $P$ and $Q$ to be each resolved into two, one parallel and the other perpendicular to the arm, it is manifest that in the latter portions of $P$ and $Q$ only is there any tendency to twist the lever; and these latter portions are respectively $P \sin CAO$, $Q \sin CBO$, hence

$$P \sin CAO \times AO = Q \sin CBO \times BO,$$

$$\text{or } P \,.\, OD = Q \,.\, OE,$$

as was proved before.

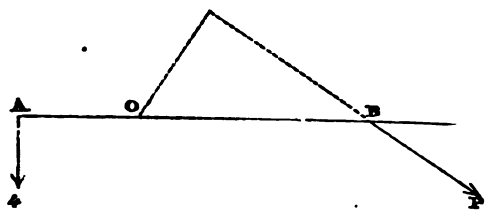
Hence we conclude, that in all cases two forces, acting in the same plane, on a rigid body, one point of which is fixed, will produce equilibrium when their moments about the fixed point are equal; for it is manifest, that the reasoning applied to the case of the simple lever, is applicable to a rigid body of any shape. This is called the *principle of the lever*.

Ex. 1. Two weights, of 3 and 4 lbs. respectively, balance on the extremities of a lever, the length of which is 6 feet: find the fulcrum.

Let $x$ be its distance from that extremity at which the weight of 3 lbs. is suspended; then $6 - x$ is its distance from the other;

$$\therefore\ 3x = 4(6 - x).$$
$$7x = 24,$$
$$x = \frac{24}{7} = 3\tfrac{3}{7} \text{ feet.}$$

Ex. 2. A weight of 4 lbs. is suspended from a straight lever, at a distance of 2 feet from the fulcrum; determine the force, which, acting at an angle of $30^0$ with the lever, and at a distance of 3 feet from the fulcrum, will produce equilibrium.



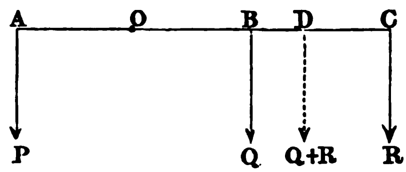

Let $P$ be the required force.

Then we have

$$P \times 3 \sin 30^0 = 4 \times 2,$$
$$P = \frac{2}{3} \times 8 = \frac{16}{3} = 5\tfrac{1}{3} \text{ lbs.}$$

18. The principle of the lever may be extended to any number of forces.

For let $AC$ be a straight lever moveable about the fulcrum $O$, and let forces be applied at $A$, $B$ and $C$. Conceive each of these forces to be resolved into two, one parallel to $AC$, the other perpendicular to it; the former parts will have no tendency to twist the lever, and we may therefore confine our attention wholly to the latter, which call $P$, $Q$, and $R$.




Now $Q$ and $R$ acting at $B$ and $C$ are equivalent to a single force $Q + R$ acting at some point $D$, such that

$$Q \,.\, BD = R \,.\, CD;$$

this follows from Art. 17. And therefore, in order that there may be equilibrium, we must have

$$P \,.\, AO = (Q + R)\, OD.$$

But $$Q \,.\, OD = Q \,.\, OB + Q \,.\, BD,$$

$$R \,.\, OD = R \,.\, OC - R \,.\, CD = R \,.\, OC - Q \,.\, BD;$$

$$\therefore P \,.\, AO = Q \,.\, OB + R \,.\, OC,$$

or the moment of $P$ = sum of the moments of $Q$ and $R$. If we estimate moments as positive or negative according as they tend to twist the lever in one direction or the other, we may then say that three forces will produce equilibrium when the *algebraical sum* of their moments* about the fulcrum is zero.

And the same method is applicable to any number of forces.

Also the same thing will hold of any forces acting in the same plane on a rigid body, one point of which is fixed; hence we may say that such forces will be in equilibrium when the algebraical sum of their moments about the fixed point is zero†.

## ON THE ACTION AND REACTION OF SMOOTH SURFACES IN CONTACT.

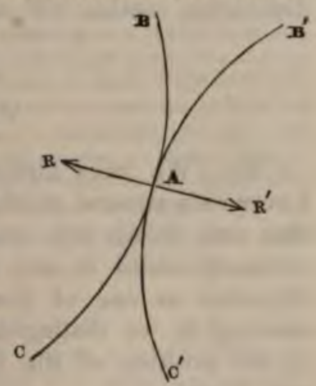
19. When two bodies are pressed together by the action of any forces, each will exert upon the other a certain force; if we call the force exerted by one body its *action*, we may call that of the other upon the first the *reaction*, and it is evident, upon consideration, that these must be equal in intensity and opposite in direction. For if a person presses

* It is manifest that $P \,.\, AO$ is the moment of the force applied at $A$. For in the figure of Art. 17, the moment of $P$ is $P \,.\, OD$, which $= P \,.\, AO \cos AOD = AO \times$ resolved part of $P$ perpendicular to $OA$.

† Hence it is not difficult to conclude the conditions of equilibrium for any rigid body, acted upon by forces the directions of which lie in the same or in parallel planes. For so far as *translation* is concerned, it will be sufficient that the sums of the forces resolved in any two directions at right angles to each other should be zero, exactly as if they acted parallel to their own directions upon a single particle; and so far as *twisting* is concerned, it will be necessary and sufficient that the forces should be such as not to twist the body about any point supposed to be fixed, that is, the algebraical sum of the moments about any point must be zero. The point about which the moments are estimated need not be in the body, since we may suppose the body rigidly connected with any point, and about such point there must be no tendency to twist.

his finger upon the table it is manifest that the table returns a pressure equal to, because it is the effect of, the pressure of the finger; and so in other cases.

But what will be the direction of this mutual pressure? We shall consider only the case of the action of two bodies smooth and lying all in one plane. Let $BAC$, $B'AC'$ be two such bodies, touching at $A$, and let $R$ be the mutual pressure. Then since the surfaces touch at $A$, they have a common tangent at that point, and therefore a common normal. Now by calling a body *smooth*, we mean to assert that there is no tendency in the constitution of the body to prevent motion along its surface, consequently no part of the mutual pressure of two smooth surfaces can be along the surface or along the common tangent, that is, *the whole must be in the direction of the common normal.*




In the case of a particle pressing upon a curve, we must consider that the pressure is in the direction of the normal to the curve; and when resting on a plane, the pressure is perpendicular to the plane.

We may observe here, that in treating statical problems which involve more than one body, we usually consider what action and reaction will exist between each two of the bodies, and having denoted them by certain symbols, as $P$, $Q$, $R$, &c., we consider the equilibrium of each body separately. An example of this will be found in the problem of the wedge (Art. 33).

## ON THE MECHANICAL POWERS.

The Mechanical Powers are the elementary forms of all machines, and may be considered as simple devices for enabling a smaller force, usually called *the Power* ($P$), to be in equilibrium with a larger force, usually called *the Weight*

($W$). They may be thus enumerated :—the Lever, the Wheel and Axle, the Toothed Wheel, the Pully, the Inclined Plane, the Wedge, and the Screw. In the following articles the ratio of $P$ to $W$ will be investigated in these several cases; if the ratio be greater than that which we determine, motion will ensue.

### (1) *The Lever.*

20. We have already considered the principle of the Lever as a general mechanical principle, and we have shewn that two forces will balance about a fulcrum when their moments about it are equal; but the lever may also be regarded as one of the Mechanical Powers, and so considering it we distinguish three kinds of lever, according to the position of the Fulcrum with respect to the Power and Weight.

The first has the fulcrum between the power and the weight. In this case any amount of mechanical advantage may be gained, by making the arm upon which the power acts sufficiently long. A crow-bar used to lift great weights, a poker, a pair of scissors, are examples. Let us examine one of these; in the poker, the coals are the weight, the bar of the fire-place the fulcrum, the force exerted by the hand the power.

The second kind of lever has the weight between the fulcrum and the power. The oar of a boat is an example, in which the water forms the fulcrum, the resistance of the boat applied at the rowlock the weight, and the power is applied by the hand of the rower.

The third kind has the point of application of the power between the fulcrum and the weight. The most interesting example is the human arm, when applied to lift a weight by turning about the elbow; here the fulcrum is the elbow, and the power is applied at the wrist by means of sinews, which exert a force when the muscles of the arm contract.

The mechanical conditions of each of these three classes may be thus expressed. If $a$ represent the arm at which the

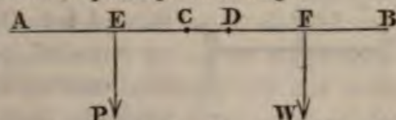
power acts, $b$ that at which the weight acts, then (supposing the power to act in a direction perpendicular to the arm)

$$Pa = Wb,$$

$$\text{or } \frac{P}{W} = \frac{b}{a} *.$$

* The condition of equilibrium of two weights balancing on a straight lever may be investigated without assuming the rules for the resolution and composition of forces as follows. The demonstration depends upon the following Lemma: *The statical effect of a uniform heavy rod is the same as if it be supposed to be collected at its middle point.* The truth of which Lemma is apparent from the consideration that such a rod would manifestly balance about its middle point, and therefore an upward pressure applied there equal to its weight would support it, and thus shew that the statical effect of the rod is to produce a downward pressure at that point equal to its weight.

This being premised, let $AB$ be a heavy uniform rod equal in weight to the sum of two given weights, $P$ and $W$; then the rod $AB$ balances about its middle point $C$.

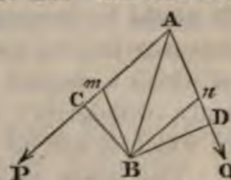



Divide $AB$ in $D$, so that $AD : DB :: P : W$, and let $E$ be the middle point of $AD$, $F$ of $DB$; then $AD$ or $P$ may be conceived to be collected at $E$ and $BD$ or $W$ at $F$. Consequently $P$ acting at $E$ will balance about $C$, $W$ acting at $F$.

But $$EC = AC - AE = BC - ED = DB - EC;$$
$$\therefore DB = 2EC,$$
$$\text{similarly } AD = 2CF;$$
$$\therefore P : W :: AD : DB,$$
$$:: CF : EC;$$
$$\text{or } P.EC = W.CF.$$

Upon the doctrine of the lever thus demonstrated it is possible to construct a complete system of statics: the steps are as follows. From the preceding proposition it is easy to conclude that two forces acting at the extremities of the arms of any lever will produce equilibrium when their moments are equal and tending to twist in opposite ways. And this being premised we can demonstrate the parallelogram of forces.

Let $Am$, $An$ represent in magnitude and direction two forces $P$ and $Q$ acting at the point $A$: complete the parallelogram $Am\,Bn$ and draw $AB$. Also draw $BC$, $BD$ perpendicular to $Am$, $An$ produced. Now suppose $AB$ to be a lever moveable about $B$ and acted on by the forces $P$ and $Q$ at $A$. Then




$$\frac{P}{Q} = \frac{Am}{An} = \frac{\sin mBA}{\sin mAB} = \frac{\sin nAB}{\sin mAB} = \frac{BD}{BC},$$
$$\text{or } P.BC = Q.BD;$$

therefore the forces $P$ and $Q$ would keep the lever at rest.

And since the resultant of $P$ and $Q$ would produce the same effect as $P$ and $Q$ together, it also acting at $A$ would keep the lever at rest. But no single force acting at $A$ can keep the lever at rest, unless it act in the direction $AB$; consequently, $AB$ is the *direction* of the resultant.

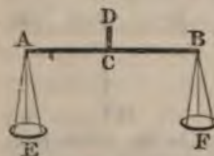
That $AB$ represents the resultant in *magnitude* must be proved as in Art. 8.

Having demonstrated the fundamental proposition in this manner, the rest of the system would be the same as in that adopted in the text.

Hence mechanical advantage is gained or not, according as $a$ is greater or less than $b$. It will be seen that in the first kind of lever advantage may or may not be gained, in the second it is always gained, in the third it is never gained. The human arm therefore acts at a mechanical disadvantage, but this is far more than compensated by the superior agility and neatness which result from its actual construction.

The weighing machines in ordinary use are applications of the principle of the lever. The simplest is the common balance.

Let $AB$ be a rigid rod, $CD$ a small rigid piece attached to its middle point and perpendicular to it, and let $D$ be fixed; $E$, $F$ two scales or pans of equal weight depending from the extremities $A$ and $B$. Then it is evident that if the scales be equally loaded, the *beam* $AB$ will be horizontal, if not, that the more heavily loaded scale will cause the extremity to which it is attached to preponderate. For the sake of clearness of explanation we have spoken of $CD$ as a small piece attached to $AB$; in reality $D$ is merely the point of suspension of the beam to the extremities of which the scales are attached.




The preceding explanation represents the balance in its simplest form, and exhibits its principles: in practice many modifications and additional contrivances must be introduced; much skill has been expended upon the construction of balances, and great delicacy has been obtained. It would be beyond the scope of this book to describe all the features in the construction of firstrate balances, by means of which a degree of accuracy has been arrived at, which is truly wonderful: there are however two or three points to which it will be desirable to call attention.

The beam should be suspended by means of a knife-edge, that is, a projecting metallic edge transverse to its length, which rests upon a plate of agate or other hard substance. The chains which support the scales should be suspended from the extremities of the beam in the same manner.

The point of support of the beam should be at equal distances from the points of suspension of the scales; and when the balance is not loaded the beam should be horizontal.

To test the accuracy of a balance, first ascertain that the beam is horizontal when the balance is not loaded; then place two weights in the scales such that the beams shall be horizontal; lastly, change these weights into opposite scales, if the beam still remain horizontal the balance is a true one.

The chief requisite of a good balance is what is termed *sensibility;* that is to say, if two weights which are very nearly equal be placed in the scales, the beam should vary *sensibly* from its horizontal position. In order to produce this result two conditions should be satisfied; (1) the point of support of the beam and the points of suspension of the scales should be in the same straight line; the consequence of this will be that two equal weights in the scales will produce a resultant through the point of support, they will therefore have no effect whatever in twisting the beam, and the deviation from horizontality will be the same for a given *difference* of weights however great the weights themselves may be; (2) the point of support should be very near the centre of gravity of the beam, and a little above it; the nearer these two points are to each other the greater will be the sensibility, for the weight of the beam acting at its centre of gravity must be in equilibrium with the small difference of the weights acting at one end of the beam, and this difference of the weights will act at a greater mechanical advantage the nearer the centre of gravity of the beam is to the fulcrum.

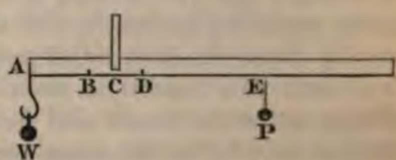
If the sensibility of a balance be very great the addition of a small weight to either scale will cause the beam to oscillate, and some time will elapse before it attains its position of equilibrium; on this account the beam is sometimes furnished with a pointer and a graduated arc of a circle; if the pointer oscillates through equal arcs on opposite sides of the point which corresponds to horizontality, we may be satisfied that the scales are equally loaded,

without waiting to ascertain whether the beam will ultimately rest in a horizontal position.

Another kind of weighing machine is the steelyard. It may be described as a lever having unequal arms, and so arranged that one weight may be made to weigh a variety of others by sliding it upon the longer arm of the lever, and so changing its distance from the fulcrum.

Prop. *To graduate the common steelyard.*

Let $C$ be the fulcrum, $W$ the subtance to be weighed hanging at the extremity $A$, $P$ the moveable weight. Now if the weights $W$ and $P$ were removed, the longer arm of the steelyard which is that upon which $P$ hangs would preponderate; suppose then that $B$ is a point such that $P$ hanging from it would keep the steelyard in a horizontal position, and take $CD = BC$, then the moment about $C$ produced by the weight of the steelyard itself is equivalent to the moment of $P$ hanging from $D$.




Now let $W$ hang from $A$, and $P$ from any point $E$, then for equilibrium we must have

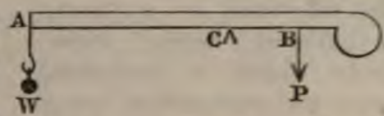
$$W \,.\, AC = P \,.\, CD + P \,.\, CE = P \,.\, BE;$$

$$\therefore\ BE = \frac{W}{P} \,.\, AC.$$

Suppose that $P = 1$lb., and make $W$ successively equal to 1lb., 2lbs., 3lbs.,...then the values of $BE$ will be $AC$, $2AC$, $3AC$,...and these distances must be set off, measuring from $B$, and the points so determined marked 1lb., 2lbs., 3lbs....

There are several varieties of the steelyard; one may be mentioned, which differs from the preceding instrument in this, that the weight is fixed and the fulcrum moveable instead of the reverse. We shall see that this makes an important difference in the mode of graduation.

PROP. *To graduate the Danish Steelyard.*




Let $B$ be the point on which the instrument would balance if no weight were suspended at $A$; and when the weight $W$ is suspended at $A$, let $C$ be the place of the fulcrum; also let $P$ be the entire weight of the instrument, which will have the effect of producing a pressure $P$ at $B$. Then for equilibrium we must have

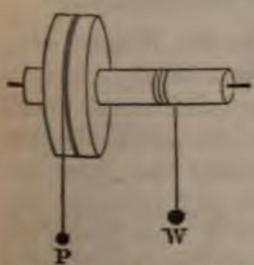
$$W . AC = P . BC = P(AB - AC);$$

$$\therefore AC = \frac{P}{W+P} . AB.$$

Hence making $W = 1$lb., 2lbs., 3lbs.,...successively, we shall be able to mark upon the steelyard the corresponding positions of the fulcrum.

It may be remarked that whereas the distances from the point $A$ of the successive marks of graduation, corresponding to equal increments of $W$, in the common steelyard form an arithmetical progression, in the Danish steelyard they form an harmonical.

## (2) *The Wheel and Axle.*




21. This machine consists of two cylinders having their axes coincident, the two cylinders forming one rigid piece, or being cut from the same piece; the larger is called the wheel, the smaller the axle. The cord by which the weight is suspended is fastened to the axle and coiled round it; the power acts, sometimes by a cord coiled round the wheel, sometimes by handspikes, as in the *capstan*, sometimes by handles, as in the *windlass.*

The *windlass* is used for such purposes as that of raising an anchor. It may be described as a strong cylindrical beam, moveable about a horizontal axis, the extremities being inserted into two strong upright pieces in which they are capable of turning freely. One end of a rope is coiled partially


round the windlass, and to the other end is attached the anchor or the weight to be raised; a number of apertures are made in the windlass perpendicular to its axis, and in these are inserted short bars called *handspikes;* by means of these it is evident that the windlass may be made to revolve, and when by its revolution a handspike is brought inconveniently low it is taken out and reinserted in a more convenient place. The windlass in the figure is represented with fixed bars, instead of handspikes, which in some applications of the machine is a more convenient arrangement.

Some inconvenience arises from the necessity of changing the position of the handspikes; this is avoided in the *capstan,* the principle of which is the same as that of the windlass, but the axis is vertical, and a person may therefore by moving his own position cause the capstan to revolve without changing the point of insertion of the handspike.

22. *To find the ratio of the Power to the Weight, when there is equilibrium upon the Wheel and Axle.*

Let $AB$, $CD$ be the wheel and axle having the common centre $O$; $P$ and $W$ the power and weight, supposed to act by strings at the circumference of the wheel and axle respectively.


For simplicity's sake $P$, $W$ and the arms at which they act are represented in the figure as in the same plane.

From the common centre $O$ draw $OA$, $OD$ to the points at which the cords supporting $P$ and $W$ touch the circumferences of the wheel and axle respectively: these lines will be perpendicular to the directions in which $P$ and $W$ act; hence, by the principle of the lever,

$$P \cdot AO = W \cdot OD,$$

$$\text{or } \frac{P}{W} = \frac{OD}{AO} = \frac{\text{radius of axle}}{\text{radius of wheel}}.$$

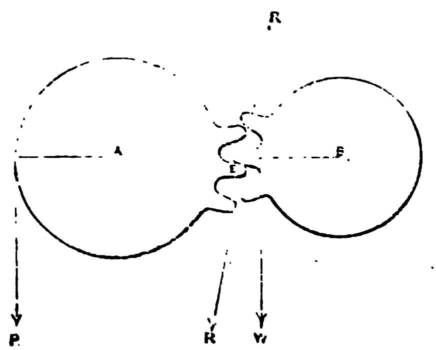
It is evident that, by increasing the radius of the wheel, any amount of mechanical advantage may be gained. It will also be seen that the principle of the wheel and axle is merely that of the lever; the peculiar advantage of the wheel and axle being this, that an endless series of levers (so to speak) are brought into play, which is essential to the practical use of the lever, when applied to such purposes as raising a bucket in a well, heaving an anchor, or the like.

### (3) *Toothed Wheels.*

23. One wheel may be made to act upon, or as it is called to *drive*, another by indenting the surface of each with teeth, and fixing the centres at such a distance from each other that the teeth come successively into contact. The proper form for the teeth of such wheels is a question of much complexity, which will not be entered upon here; we shall only investigate in general the relation of $P$ to $W$, when there is equilibrium.

24. *To find the ratio of the Power to the Weight in Toothed Wheels.*

Let $A$, $B$ be the centres of the wheels, on the circumference of which the teeth are arranged, and suppose for simplicity's sake that $P$ and $W$ act at the circumferences of the wheels, and that the radii of the same are $rr'$ respectively.



Also let two of the teeth be in contact at $E$, and let $R$ be the mutual pressure of the teeth in contact, which acts in the direction of the common normal to the surfaces of the teeth; and let $pp'$ be the lengths of the perpendiculars from $A$ and $B$ respectively on the common normal.

Then the wheel, of which the centre is $A$, may be supposed to be kept in equilibrium by the forces $P$ and $R$ tending to twist it in opposite ways; hence by the principle of the lever,

$$P \cdot r = R \cdot p.$$

Similarly, for the equilibrium of the other wheel,

$$W \cdot r' = R \cdot p'.$$

Hence $$\frac{Pr}{Wr'} = \frac{p}{p'},$$

or $$\frac{P}{W} = \frac{pr'}{p'r}.$$

(4) *The Pully.*

25. The pully, in its simplest form, consists of a wheel, capable of turning about its axis, which may be either fixed

Fig. I.

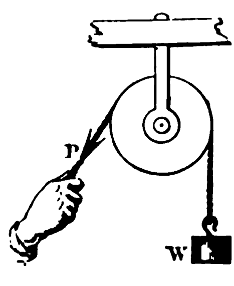
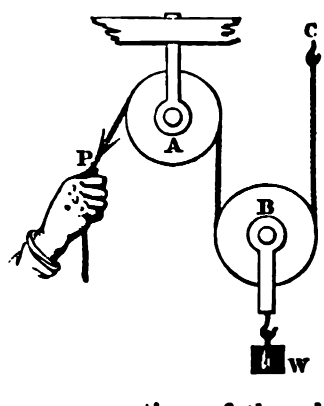



Fig. II.

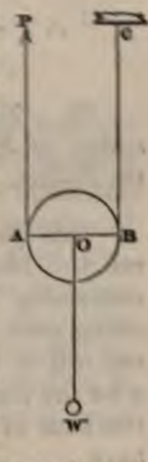



or moveable. A cord passes over a portion of the circumference; if the axis of the pully is fixed (Fig. I.) its only effect

is to change the direction of the force exerted by the cord, but if it is moveable (Fig. III.) a mechanical advantage may be gained, as we shall see immediately. Combinations of pullies may be made in endless variety; we shall here consider only the simple moveable pully, and some of the more ordinary combinations.

In what follows, no account will be taken of the weights of the pullies themselves, but the principles may easily be extended to that case. Also in practice there will be a considerable amount of friction, when $P$ is on the point of descending, but this is neglected for the sake of greater simplicity.

26. *To find the ratio of the Power to the Weight in the simple moveable Pully.*



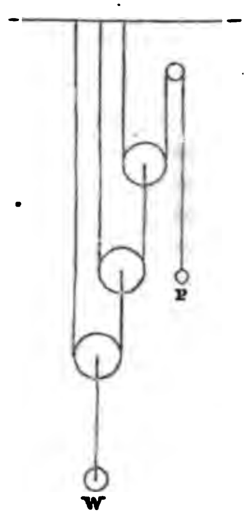

Let $O$ be the centre of the pully, which is supported by a cord passing under it and attached to some fixed point $C$ at one end, and stretched by the force $P$ at the other. Suppose the weight $W$ to be suspended from the centre $O$.

Join the points $A$, $B$, at which the contact of the cord with the pully commences, by a straight line $AB$, which will pass through the centre $O$. Then we may consider the mechanical conditions of the problem to be the same as those of a lever $AB$, kept in equilibrium about the fulcrum $O$ by the force $P$ at $A$ and the tension of the string at $B$. But the tension of the string must be the same throughout, and is therefore equal to $P$. Hence the force at each end of the lever is $P$, and the resultant of these two parallel forces $2P$. But this resultant supports $W$;

$$\therefore 2P = W,$$

$$\text{or } \frac{P}{W} = \frac{1}{2}.$$

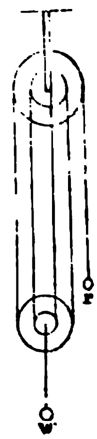
27. *To find the ratio of the Power to the Weight, in a system of Pullies in which each pully hangs by a separate string.* (First system of Pullies.)



This system of pullies is represented in the figure. Suppose there are $n$ pullies; then the tension of the string passing under the first $= \frac{W}{2}$ (by the property of the simple pully). The tension of the string passing under the second $= \frac{W}{2^2}$, and so on. That of the string under the last pully $= \frac{W}{2^n}$; but this must be equivalent to the power $P$;

$$\therefore P = \frac{W}{2^n}, \qquad \text{or } \frac{P}{W} = \frac{1}{2^n}.$$

28. *To find the ratio of the Power to the Weight, in a system of Pullies in which the same string passes round all the Pullies.* (Second system of Pullies.)



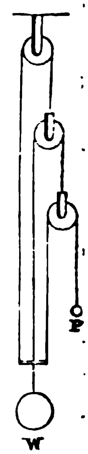
This system is represented in the figure. There are two blocks, the lower one moveable, and each containing a number of pullies. Since the same string goes round all the pullies, the tension throughout will be the same, and equal to the power $P$. Let $n$ be the number of strings at the lower block, then the sum of their tensions will be $nP$, and we shall have

$$nP = W,$$

$$\text{or } \frac{P}{W} = \frac{1}{n}.*$$

* A little consideration will shew that if the wheels in this system of pullies be equal, and if the system be put in motion by the descent of $P$, the parts of the rope which pass in the same time over the wheels in the lower block are in the proportion of the numbers 1, 3, 5......, whilst the parts which pass over the wheels in the upper are in the proportion

29. *To find the ratio of the Power to the Weight, in a system of Pullies in which all the strings are attached to the Weight.* (Third system of Pullies.)




The figure represents the system. The tension of the string by which $P$ hangs is $P$; that of the next $= 2P$ (by the property of the simple pully;) that of the next $2^2P$, and so on. Let there be $n$ strings, then the tension of the last $= 2^{n-1}P$, and the sum of all the tensions

$$= (1 + 2 + 2^2 + \dots\dots\dots\dots + 2^{n-1})\, P = W;$$

$$\text{or } \frac{P}{W} = \frac{1}{1 + 2 + 2^2 + \dots\dots\dots\dots + 2^{n-1}} = \frac{1}{2^n - 1}.$$

### (5) *The Inclined Plane.*

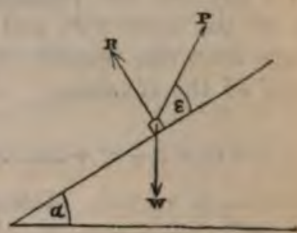
30. By the inclined plane is meant a plane inclined to the horizon, and the problem is to find the force necessary to prevent a body placed upon it from sliding down under the action of its own weight. The plane is supposed smooth, and therefore, for reasons already explained (Art. 19, page 222), will exert a pressure on the body in the direction of a line perpendicular to its surface. We shall have to apply here the general principles of equilibrium before deduced, viz. that the sum of the forces acting on the weight, resolved in any two directions perpendicular to each other, must sepa-

of 2, 4, 6,......So that while the first wheel below revolves once, the first wheel above revolves twice, the second wheel below three times, and so on. If, however, the wheels differed in size in proportion to the quantity of rope which must pass over them, they would revolve in the same time. Thus if the radii of the wheels in the lower block be made in the proportion of the numbers 1, 3, 5......and those in the upper in the proportion of 2, 4, 6......, the wheels would all revolve in the same time; and this being the case we may observe further that such wheels might be cut in the faces of two solid pieces. This is the principle of *White's pully*, a machine extremely ingenious in its conception and presenting considerable advantages when accurately constructed but practically little used.

rately vanish: the two equations furnished by these conditions will enable us to determine not only the ratio of the power to the weight, but also the pressure of the weight on the plane.

31. *To find the ratio of the Power to the Weight, when there is equilibrium on the inclined Plane.*

Let $\alpha$ be the inclination of the plane to the horizon; $R$ the pressure of the plane on the weight, which will be perpendicular to the plane; and let $\epsilon$ be the angle which the direction of $P$ makes with the plane. Then, resolving the forces parallel and perpendicular to the plane, we have




$$P \cos \epsilon - W \sin \alpha = 0 \dots\dots\dots (1),$$

$$R + P \sin \epsilon - W \cos \alpha = 0 \dots\dots\dots (2).$$

Hence $$\frac{P}{W} = \frac{\sin \alpha}{\cos \epsilon}.$$

COR. 1. If the power acts parallel to the plane, $\epsilon = 0$, and

$$\frac{P}{W} = \sin \alpha.$$

COR. 2. From the preceding corollary it will be easily seen that a uniform cord, part of which rests upon an inclined plane, and the remainder hangs freely from the upper extremity of the plane, will be in equilibrium, provided the two ends of the cord are in the same horizontal line. For let $a$ be the portion which rests on the plane, $b$ the portion which hangs vertically, then $b = a \sin \alpha$; but the force $P$ acting along the plane, which is due to the weight of the vertical portion, is proportional to $b$, and the weight sustained by the plane or $W$ is proportional to $a$, $\therefore P = W \sin \alpha$, which is the condition of equilibrium.

COR. 3. More generally, there will be equilibrium when a uniform cord rests upon two inclined planes having a com-

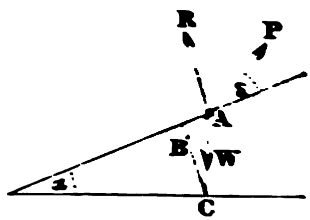
mon vertex, provided the extremities of the axis are in the same horizontal line*.

Cor. 4. If it is required to find the pressure $R$ we have, multiplying (1) by $\sin \epsilon$ and (2) by $\cos \epsilon$, and subtracting,

$$R \cos \epsilon = W \cos (\alpha - \epsilon).$$

If $\epsilon = 0$, $\qquad R = W \cos \alpha$.

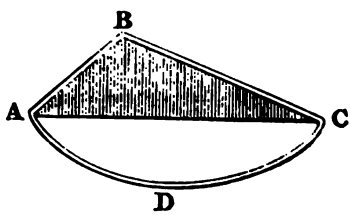
31. (bis). The results of the preceding article may also be obtained as follows:




Let $a$, $\epsilon$, $R$ represent the same quantities as before. Let $A$ be the point of the plane at which the weight rests; draw $AC$ vertical, and from $C$ draw $CB$ in a direction perpendicular to the inclined plane to meet the line of $P$'s action in $B$. Then the sides of the triangle $ABC$ being parallel to the directions of the forces $P$, $R$, $W$, may be taken to represent those forces. (Art. 11.) Hence

* The conditions of equilibrium on the inclined plane, from which the whole theory of the resolution of forces may be deduced, were determined with singular ingenuity by Stevinus of Bruges, in 1586. The substance of his reasoning was as follows:

Let $ABC$ be a triangular board forming two opposite inclined planes. Suppose a loop of uniform cord to be suspended from it, as in the figure, so as to rest upon the two sides $AB$, $CB$, and depend below in the symmetrical curve $ADC$. Then it is manifest that there will be equilibrium.




Now the tension produced at $A$ by the portion of the cord $ADC$ must be equal to the tension produced at $C$ by the same, on account of the symmetry of the curve $ADC$. Consequently equilibrium will still subsist if we remove the portion of cord $ADC$, that is, the portions of string $AB$, $BC$ on the opposite inclined planes will be in equilibrium; which is the result already obtained, and from which it immediately follows that weights connected by a string will be in equilibrium on opposite inclined planes, when they are proportional to the lengths of the planes on which they rest, or inversely proportional to the sines of the inclinations of the planes.

$$\frac{P}{\sin C} = \frac{W}{\sin B} = \frac{R}{\sin A}.$$

But $A = 90^0 - a - \epsilon$,

$B = 90^0 + \epsilon$,

$C = a$,

$$\therefore \frac{P}{\sin a} = \frac{W}{\cos \epsilon} = \frac{R}{\cos (a + \epsilon)}.$$

### (6) *The Wedge.*

32. The wedge is a triangular prism, made of some hard substance, as steel, the edge of which is introduced between two obstacles, which it is our purpose to separate. When the edge is introduced, the wedge is driven forward by a violent blow, as from a hammer or the like, which generates an enormous force of momentary duration. We shall consider the wedge to be acted upon by a weight resting upon its head, but the principles of the investigation are applicable to all cases, in whatever manner the pressure on the wedge is produced; we shall also suppose the wedge to be isosceles, and the obstacles on opposite sides of the wedge to be exactly similar. When the wedge is driven in between two obstacles, as for instance when applied to split the trunk of a tree, the obstacles have a tendency to fly together, owing, in the instance supposed, to the tenacity of the fibres, and this is the resistance which we consider, and which corresponds to the weight supported in the Mechanical Powers already treated of. In practice there will usually be a great amount of friction between the wedge and obstacles, but this, for the sake of simplifying the mathematical investigation, we shall omit.

33. *To find the ratio of the Power to the Resistance, when an isosceles Wedge is kept in equilibrium by the pressure of two obstacles symmetrically situated with respect to it.*

Let $2P$ be the force acting on the head of the

wedge, $\alpha$ the semi-angle of the wedge. Also let $A$ be one of the points at which the wedge is in contact with the obstacle; then there will be a mutual pressure at this point between the wedge and obstacle, which will be perpendicular to the side of the wedge, and which we will call $R$. There will be a similar pressure, on the other side of the wedge, from the other obstacle.

Fig. I.

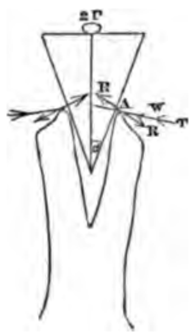
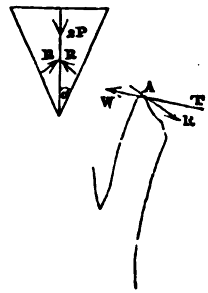



Fig. II.




Again, the point $A$ of the obstacle is acted upon by the force which we call $W$, and which forms the resistance to motion. To determine the direction in which $W$ acts, we observe that, if $A$ were made to move by the descent of the wedge, it would begin to move in some curve line, and that the tangent to that curve ($AT$) is the direction in which $W$ acts. The position of this line $AT$ is quite unknown, but if we denote the angle between it and the direction of $R$ by $i$, we may be sure that $i$ is in general small.

In Figure II. we have represented the wedge and obstacle separate from each other, in order to shew clearly the forces which respectively retain them in equilibrium. (The point $A$ will of course require a third force to keep it in equilibrium, this will be perpendicular to $AT$ and will arise from the pressure on the ground which supports the obstacle; it is omitted in the investigation, because its magnitude is a matter of no interest.)

Now the wedge is kept in equilibrium by the force $2P$ and the two forces $R$; hence, resolving in the direction of $P$, we have

$$2P = 2R \sin \alpha \quad \ldots\ldots\ldots (1).$$

Again, the point $A$ is kept in equilibrium, so far as tendency to motion in the direction $AT$ is concerned, by $W$ and the resolved part of $R$: hence

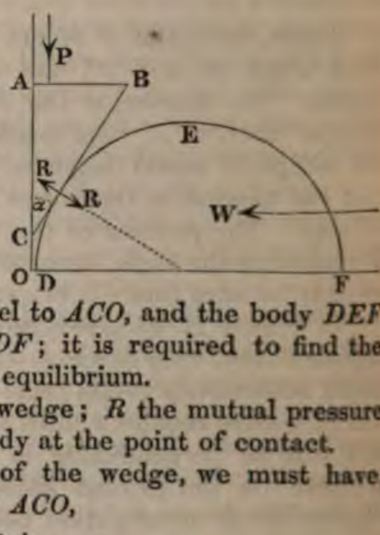
$$W = R \cos i \;\ldots\ldots\ldots (2).$$

From (1) and (2) we have

$$\frac{P}{W} = \frac{\sin \alpha}{\cos i}.$$

If $i$ be very small we have $P = W \sin \alpha$, nearly.

33 (bis). In addition to the preceding mode of considering the wedge, which necessarily introduces the unknown angle $i$, we may consider it as acting upon some other body the possible motion of which is constrained so as to be able to take place only in a certain direction; in this case no unknown angle will occur, and it is in this form only that the wedge can be regarded as applicable to the purposes of practical mechanics.




Suppose, for instance, that the wedge $ACB$ is constrained so as to be capable of motion only in the direction $ACO$, and that it acts upon the semicircular body $DEF$ which can only move along $ODF$, a line perpendicular to $ACO$; and suppose the wedge to be acted upon by a force $P$ parallel to $ACO$, and the body $DEF$ by a force $W$ parallel to $ODF$; it is required to find the ratio of $P$ to $W$ when there is equilibrium.

Let $\alpha$ be the angle of the wedge; $R$ the mutual pressure between the wedge and the body at the point of contact.

Then for the equilibrium of the wedge, we must have, resolving the forces parallel to $ACO$,

$$P = R \sin \alpha;$$

for the equilibrium of the semicircular body, resolving parallel to $ODF$, and observing that the angle which the direction of $R$ makes with $ODF$ is the angle of the wedge,

$$W = R \cos \alpha;$$

$$\therefore \frac{P}{W} = \tan \alpha.$$

## The Screw.

34. The Screw may be conceived of as an inclined plane wrapped round a cylinder, or as a cylinder having on its surface a projecting thread inclined in all parts at the same given angle to the horizon. This cylinder fits into a block pierced with an equal cylindrical aperture, on the inner surface of which is cut a groove the exact counterpart of the thread on the screw; hence we can cause the screw to enter the block only by making it revolve about its axis. Suppose the axis of the screw to be vertical, and a weight $W$ to be placed upon it, then the screw would descend unless prevented from doing so by some other force; this force we suppose to be supplied by a power $P$ acting in a horizontal direction at the extremity of an arm of given length: this is nearly the mode in which the screw is actually applied to certain mechanical purposes, as to the bookbinder's press, and the like. In practice the friction between the thread of the screw and the block will generally be considerable, but for the sake of simplicity we shall consider everything to be perfectly smooth.

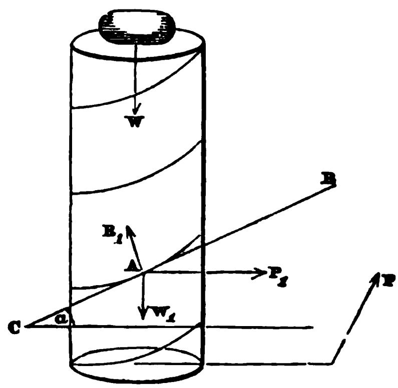
In practice also the thread of the screw will be a projection upon the surface of the cylinder of considerable magnitude, (as exhibited in the figure at the foot of page 243,) and the form of the thread will be determined by a variety of circumstances, such as the substance of which the screw is made, the purpose for which it is intended, and the like; but in our investigations we shall consider, for simplicity's sake, that the thread is indefinitely small, and that the pressure upon it is exerted in a plane touching the cylinder. A very good notion of the kind of screw, with which we shall be concerned, may be gained by cutting a right-angled triangle in paper, and wrapping it upon a cylinder, the hypothenuse being thus made to form the thread.

35. *To find the ratio of the Power to the Weight in the Screw.*

Let the power $P$ act at an arm $a$, and let $r$ be the radius of the cylinder, $\alpha$ the inclination of the thread to the horizon.

Consider the equilibrium of any point $A$ of the thread; suppose the portion of the thread on each side of $A$ to

be unwrapped, so as to assume the position of the straight line $BC$, inclined at an angle $a$ to the horizon; then we may consider the point $A$ as supported on a plane of inclination $a$, and acted upon by the pressure on the plane $R_1$, a horizontal force $P_1$, and a portion of the weight $W$, which we will call $W_1$; hence, resolving the forces along the plane, we must have




$$W_1 \sin a = P_1 \cos a;$$

in like manner, if we take another point of the thread, we must have

$$W_2 \sin a = P_2 \cos a,$$

and so on. Hence, taking into account all points of the thread, and adding together the equations, we shall have

$$(W_1 + W_2 + W_3 + \ldots) \sin a = (P_1 + P_2 + P_3 + \ldots) \cos a.$$

But $W_1 + W_2 + W_3 + \ldots =$ the whole weight supported $= W$. Also, $P_1 + P_2 + P_3 + \ldots =$ the whole horizontal force supposed to act at the circumference of the cylinder, i. e. at an arm $r$.

But the horizontal pressure is caused by $P$ acting at an arm $a$; hence, by the principle of the lever,

$$(P_1 + P_2 + P_3 \ldots)\, r = Pa;$$

$$\therefore\ W \sin a = \frac{Pa}{r} \cos a,$$

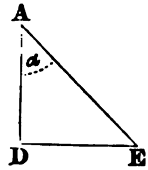
$$\text{or } \frac{P}{W} = \frac{r}{a} \tan \alpha.$$

We may put this result in a rather different form: thus,

$$\frac{P}{W} = \frac{2\pi r \tan \alpha}{2\pi a},$$

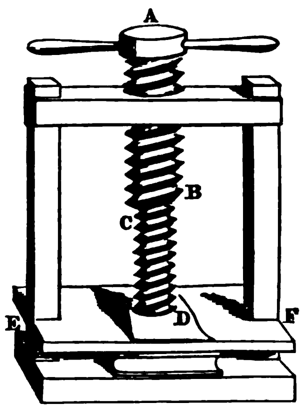
$$= \frac{\text{vertical distance between two threads}}{\text{circumference of circle described by } P}^{*}.$$

35 (bis). The first part of the preceding article may be replaced by the following:




Draw $AD$ vertical, $DE$ horizontal, $AE$ in a direction perpendicular to the thread of the screw; then the sides of the triangle $ADE$, being parallel to the directions of the weight supported by the screw, the horizontal pressure produced by the power $P$, and the resultant of the pressures on the thread, may be taken to represent those forces;

$$\therefore \text{ the horizontal pressure} = W\frac{DE}{AD} = W \tan \alpha.$$




* It will be seen from this expression that in order to increase the efficiency of the screw we must either increase the length of the lever at which $P$ acts, or diminish the distance between the threads of the screw. The former method is limited by the unwieldy character which would be given to the machine if the arm of the lever were made very long; if the latter be resorted to, the too great diminution of the thread will so weaken it that a slight resistance will tear it from the cylinder. These difficulties are obviated by *Hunter's Screw*, which consists of two screws, one coarser than the other, the finer screw usually fitting into the cylinder, the surface of which carries the coarser thread. Thus when the working point of the machine is urged forward by the coarser screw it is drawn back by the finer, and the formula for the machine will be found to be,

$$\frac{P}{W} = \frac{\text{difference of vertical distances between the threads}}{\text{circumference of circle described by } P}.$$

And it is manifest that this ratio may be diminished to any extent by making the vertical distance between the threads in the two screws nearly equal, each screw being as coarse as we please.

But the horizontal pressure $= P\dfrac{a}{r}$.

$$\therefore \frac{P}{W} = \frac{r}{a}\tan\alpha.$$

## ON FRICTION.

36. When we attempt to make the surface of a body move upon that of another, with which it is kept in contact by pressure, there is in general a resistance to motion, which is frequently sufficient to prevent it altogether; the force of resistance is called *friction*.

Friction at any point of a surface always acts in the direction exactly contrary to that in which the point tends to move. Hence, when a particle is placed on a rough plane, the line in which the friction acts will lie in the plane. Also, if a body on an inclined plane, and under the action of any force, is on the point of *ascending*, the force of friction acts *downwards*; but if the weight of the body is so great that it is on the point of *descending*, the action of friction is *upwards*.

It has been shewn by experiment, that when a body is on the point of moving on the surface of another, and is only prevented from doing so by friction, then the force of friction bears to the normal pressure between the two bodies a ratio which depends only upon the constitution of the two bodies. In fact, let $R$ be the normal pressure, then the friction will be expressed by $\mu R$, where $\mu$ is a quantity depending not on the pressure nor on the extent of the surfaces in contact, but only on the nature of the bodies; it has, for instance, a certain definite value for metal and wood, and so on. The quantity $\mu$ is called the *coefficient of friction**.

The fact that the friction between two bodies is independent of the extent of surface in contact may appear at first sight paradoxical; a very simple consideration however will

* The friction spoken of in this article may be termed *statical* friction; when one rough body actually moves upon the surface of another a different kind of friction is brought into play, which may be called *dynamical* friction. This latter is shewn by experiment to be proportional to the pressure, and independent of the extent of surface in contact and of the velocity of the moving body.

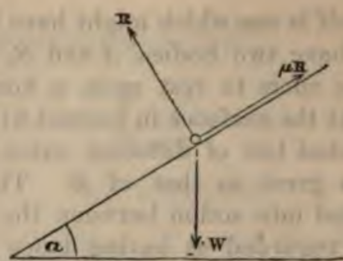
shew that the result is one which might have been anticipated. Suppose that we have two bodies, $A$ and $B$, of equal weight, and that we cause them to rest upon a horizontal plane in such a manner that the surfaces in contact with the plane shall be of the same kind but of different extent, that of $A$ for example twice as great as that of $B$. Then if we cause friction to be called into action between the bodies and the plane, $A$ may be regarded as having twice as many points for the exertion of friction as $B$: but on the other hand the weight of $A$ is distributed over twice as many points as that of $B$, and therefore the weight supported by each point and the friction produced at each point of $A$ may be considered to be half that at each point of $B$. On the whole therefore we have in $A$ twice as many points for friction as in $B$, and half the friction at each point, and so the same amount of friction upon the whole.

From what has been seen it will appear, that in problems involving the pressure of one body on another, there will not be a greater number of unknown forces involved on the supposition of the bodies in contact being rough, than there would be on the hypothesis of their being smooth, provided we consider only the limiting circumstances of equilibrium, that is, when the body is on the point of sliding.

In consequence of the force of friction, systems of bodies in nature are not obliged to fulfil those exact conditions, which would be necessary if such a force did not exist. For example, a body would not retain its position on a smooth plane unless the plane were accurately horizontal, whereas a rough plane may be considerably inclined without disturbing the equilibrium of a body upon it.

37. We shall illustrate this by finding the angle at which a rough plane may be inclined, so that a body may just rest upon it without sliding.

Let $\alpha$ be the angle of inclination of the plane; $W$ the weight of the body; $R$ the normal pressure on the plane; $\mu R$ the force of friction. Then, resolving the forces perpendicular to the plane and parallel to it, we have these equations of equilibrium:



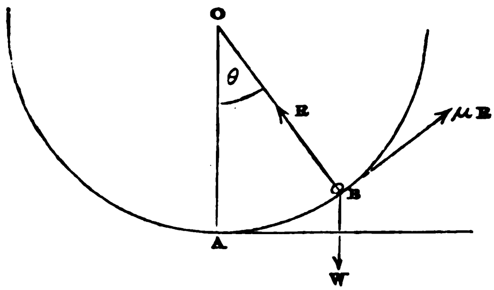

$$R = W \cos \alpha,$$
$$\mu R = W \sin \alpha;$$
$$\therefore \tan \alpha = \mu.$$

This equation determines the limiting value of the inclination of the plane, for which equilibrium is possible; for any smaller value there will be equilibrium *à fortiori.*

It will be observed, that if the condition which has been investigated be satisfied, equilibrium will subsist however great $W$ may be; and the reason is, that in whatever manner we increase $W$, in the same proportion we increase the friction upon the plane which serves to prevent $W$ from sliding. A useful application of this principle is the case of the carpenter's nail. A nail may be regarded as a wedge driven in between two obstacles; now if these obstacles press upon the sides of the wedge and the angle of the wedge be not sufficiently small, it will not remain in its place; but if the angle be small enough, then whatever be the pressure upon the sides of the wedge it cannot be pressed from its hold.

38. Again, if the interior of a hemispherical bowl is smooth, a body cannot rest in it except at the lowest point; but if it be rough, there will be certain limits within which the equilibrium will be possible; let us determine those limits.

Let $B$ be the highest position of the body possible, $A$ the bottom of the bowl, $O$ the centre; $AOB = \theta$. The direction of the pressure $R$ will pass through the centre, the friction $\mu R$ will be perpendicular to $BO$. Resolving the forces in the direction of $BO$ and perpendicular to it, we have




$$R = W \cos \theta,$$
$$\mu R = W \sin \theta;$$
$$\therefore \tan \theta = \mu.$$

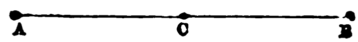
This equation determines the greatest possible angular distance of the body from the bottom of the bowl. The vertical height of the body above the lowest point $A$

$$= r(1 - \cos \theta) = r\left\{1 - \frac{1}{\sqrt{1+\mu^2}}\right\}.$$

Suppose, for instance, that $r = 1$ foot, and that $\mu = \frac{1}{4}$, which is its value for metallic surfaces, then the preceding expression becomes $1 - \frac{4}{\sqrt{17}} = .029$ of an inch, nearly.

## THE CENTRE OF GRAVITY.

39. If two equal heavy bodies $A$ and $B$ are connected by a fine rod, it is evident that the system will balance, if the middle point $C$ of the rod be supported. And this will be the case in whatever position the system is placed, because the moment of $A$, tending to twist the rod in one direction, will always be equal to the moment of $B$, tending to twist it in the opposite. The point $C$, about which $A$ and $B$ will balance in any position, is called *the centre of gravity* of the bodies.



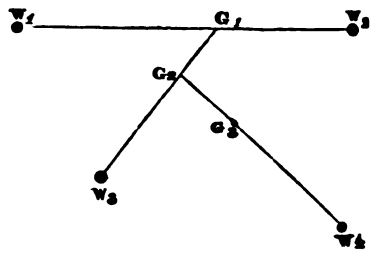

If the bodies $A$ and $B$ be unequal, and the distance $AB$ be divided in $C$, in such a manner that

$$AC : BC :: \text{weight of } B : \text{weight of } A,$$

the system will still balance in any position, if $A$ and $B$ be supposed rigidly connected with $C$, and $C$ is called their *centre of gravity* as before.

It may be shewn, that for every system of heavy particles there exists in like manner one point, and no more, such that, if it be fixed and the bodies rigidly connected with it, the system will rest in any position. This point is called *the centre of gravity of the system.*

40. *To shew that every system has a centre of gravity.*




Let $W_1$, $W_2$, $W_3$, $W_4$, .........be a system of particles, the weights of which are $W_1$, $W_2$, $W_3$, $W_4$,...... respectively, suppose $W_1$, $W_2$ joined by a rigid rod without weight, and divide the same rod in $G_1$, so that

$$W_1G_1 : W_2G_1 :: W_2 : W_1,$$

then $W_1$ and $W_2$ will balance in all positions about $G_1$, and if we suppose $G_1$ supported, the pressure upon the support will be $W_1 + W_2$.

Again, join $G_1W_3$, and divide it in $G_2$, so that

$$G_1G_2 : W_3G_2 :: W_3 : W_1 + W_2;$$

then, if we suppose the rod $W_1W_2$ to rest upon the rod $G_1W_3$ and $G_2$ to be supported, the pressure $W_1 + W_2$ at $G_1$ and $W_3$ at $W_3$ will balance about $G_2$. Hence the three bodies $W_1$, $W_2$, $W_3$, supposed rigidly connected, will balance in all positions about $G_2$.

Similarly we may find a point $G_3$ about which $W_1$, $W_2$, $W_3$, $W_4$ will balance in all positions, and so of any number of particles. Hence every system of particles has a centre of gravity.

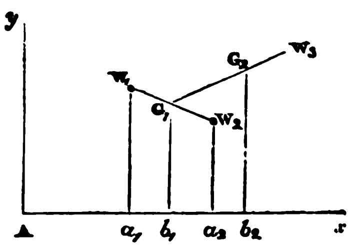
It is obvious from the method of proof that the system of particles need not lie all in one plane.

41. *A system can have only one Centre of Gravity.*

For suppose there are two, and let the system be so turned that the two centres of gravity lie in the same horizontal plane. Then the weights of the different particles of the system form a system of vertical forces, the resultant of which must pass through the centre of gravity, otherwise the system could not balance about that point; hence the said vertical resultant must pass through two points in the same horizontal plane, which is absurd. Therefore there are not two centres of gravity.

42. It is manifest from the mode by which we proved the existence of a centre of gravity, that the tendency of a system of heavy particles to produce pressure, or to cause moment about any point, is the same as that of a single particle equal in weight to that of the whole system and situated at its centre of gravity. This is sometimes expressed by saying, that we may suppose a system *collected at its centre of gravity.*

43. *To find the centre of gravity of any number of particles in the same plane.*




Let $W_1$, $W_2$, $W_3$......be the weights of the particles; in the plane in which they lie, take any two straight lines $Ax$, $Ay$, at right angles to each other, and let $h_1$, $h_2$, $h_3$.........be the distances of $W_1$, $W_2$, $W_3$...... from the line $Ax$, and $k_1$, $k_2$, $k_3$.........their distances from the line $Ay$; also let $h$, $k$ be the distances of the centre of gravity of the system from $Ax$, $Ay$ respectively; then it is evident that if we find $h$ and $k$, we shall have solved the problem.

Join $W_1$, $W_2$, and let $G_1$ be the centre of gravity of $W_1$, $W_2$; from $W_1$, $W_2$, $G_1$, draw $W_1a_1$, $W_2a_2$, and $G_1b_1$ perpendicular to $Ax$: then we have

$$W_1 \times W_1G_1 = W_2 \times W_2G_1;$$

but it is evident, from similar figures, that

$$W_1G_1 : a_1b_1 :: W_2G_1 : a_2b_1;$$

$$\therefore\ W_1 \times a_1b_1 = W_2 \times a_2b_1,$$

$$\text{or } W_1(Ab_1 - k_1) = W_2(k_2 - Ab_1);$$

$$\therefore\ Ab_1 = \frac{W_1k_1 + W_2k_2}{W_1 + W_2}.$$

If we consider another particle $W_3$, we may, in searching for the centre of gravity of the three $W_1$, $W_2$, $W_3$, suppose the two former to act together at their centre of gravity already found; hence, if $G_2$ be the centre of gravity of the three particles, and we draw $G_2b_2$ perpendicular to $Ax$, we shall have

$$Ab_2 = \frac{(W_1 + W_2)Ab_1 + W_3k_3}{W_1 + W_2 + W_3}$$

$$= \frac{W_1k_1 + W_2k_2 + W_3k_3}{W_1 + W_2 + W_3};$$

and so on for any number of particles. Hence we shall have

$$k = \frac{W_1k_1 + W_2k_2 \ldots\ldots\ldots\ldots + W_nk_n}{W_1 + W_2 \ldots\ldots\ldots + W_n};$$

and, in like manner,

$$h = \frac{W_1h_1 + W_2h_2 \ldots\ldots\ldots\ldots + W_nh_n}{W_1 + W_2 \ldots\ldots\ldots + W_n}.$$

44. *To find the centre of gravity of any number of particles not in the same plane.*

If we conceive three planes perpendicular to each other to be drawn, we shall solve the problem if we find the perpendicular distance of the centre of gravity from each of these planes. Let $h_1$, $k_1$, $l_1$, be the perpendicular distances of the particle $W_1$ from the three planes respectively, and so on of the other particles, and $h$, $k$, $l$, the perpendicular distances of the centre of gravity: then we may prove, in like manner as in the last proposition, that

$$h = \frac{W_1 h_1 + W_2 h_2 \ldots\ldots\ldots\ldots + W_n h_n}{W_1 + W_2 \ldots\ldots\ldots\ldots + W_n};$$

similarly,

$$k = \frac{W_1 k_1 + W_2 k_2 \ldots\ldots\ldots\ldots + W_n k_n}{W_1 + W_2 \ldots\ldots\ldots\ldots + W_n};$$

and so likewise,

$$l = \frac{W_1 l_1 + W_2 l_2 \ldots\ldots\ldots\ldots + W_n l_n}{W_1 + W_2 \ldots\ldots\ldots\ldots + W_n}.$$

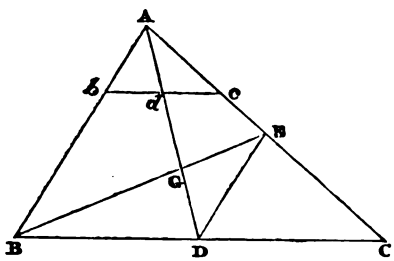
We shall now proceed to find the centre of gravity in some actual cases.

45. *To find the centre of gravity of a physical right line, or of a uniform thin rectilinear rod.*

The middle point will be the centre of gravity; for we may suppose the line to be divided into pairs of equal weights equidistant from the middle point, and the middle point will be the centre of gravity of each pair, and therefore of the whole system.

46. *To find the centre of gravity of a triangle, or of a thin lamina of matter in the form of a triangle.*

Let $ABC$ be the triangle; bisect $BC$ in $D$, and join $AD$; draw any line $bdc$ parallel to $BC$; then it is evident that this line will be bisected by $AD$ in $d$, and will therefore balance about $d$, in all positions; similarly all lines in the triangle parallel to $BC$ will balance about points in $AD$, and therefore the centre of gravity must be somewhere in $AD$.




In like manner, if we bisect $AC$ in $E$, and join $BE$, the centre of gravity must be in $BE$; hence $G$, the intersection of $AD$ and $BE$, is the centre of gravity of the triangle.

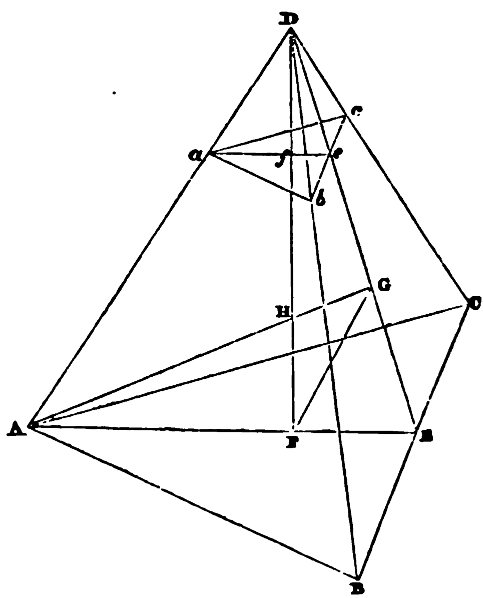
Join $DE$, which will be parallel to $AB$, (Euclid, VI. 2) then the triangles $ABG$, $DEG$ are similar.

$$\therefore AG : GD :: AB : DE$$
$$:: BC : DC$$
$$:: 2 : 1,$$
$$\text{or } AG = 2GD,$$
$$\text{and } \therefore AD = 3GD.$$

Hence, if we join an angle of a triangle with the bisection of the opposite side, the point which is two thirds of the distance down this line from the angular point is the centre of gravity of the triangle.

Obs. It is not difficult to see, that if three equal particles be placed at the angular points of the triangle $ABC$, their centre of gravity will coincide with that of the triangle $ABC$.

47. *To find the centre of gravity of a pyramid on a triangular base.*




Let $ABCD$ be the pyramid. Bisect $BC$ in $E$; join $AE$: take $EF = \frac{1}{3} AE$, and join $DF$.

Suppose the pyramid to be made up of thin slices parallel to $ABC$, and let $abc$ be one of them; let $afe$ be the line in

which it is intersected by the plane $DAE$, $e$ and $f$ lying in $bc$ and $DF$ respectively. Then, by similar triangles,

$$be : eD :: BE : ED,$$
$$\text{also } ce : eD :: CE : ED;$$
$$\therefore be : ce :: BE : CE,$$
$$\text{but } BE = CE; \therefore be = ce.$$

In like manner it may be shewn, that

$$fe : af :: FE : AF,$$
$$\text{but } AF = 2FE; \therefore af = 2fe.$$

Hence $f$ is the centre of gravity of the triangle $abc$. Similarly it will appear, that the centres of gravity of all slices of the pyramid made by planes parallel to $ABC$ lie in $DF$, and therefore the centre of gravity of the pyramid is in that line.

Similarly, if we join $DE$, take $GE = \frac{1}{3}DE$, and join $AG$, the centre of gravity will be in $AG$; therefore $H$, the intersection of $DF$ and $AG$, is the centre of gravity of the pyramid.

Now join $GF$, then by similar triangles,

$$\begin{aligned} HF : HD &:: GF : AD \\ &:: FE : AE \\ &:: 1 : 3; \end{aligned}$$
$$\therefore HF = \tfrac{1}{3}HD = \tfrac{1}{4}DF.$$

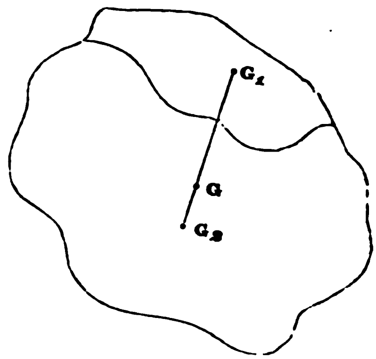
Hence, if we join the vertex of the pyramid with the centre of gravity of the base, and set off one fourth of this line from the latter point, we shall determine the centre of gravity of the pyramid.

Cor. 1. The same construction will hold for a pyramid upon a base of any form, since it may be divided into a number of pyramids on triangular bases.

Cor. 2. The centre of gravity of a solid cone will be found, by setting off one fourth of the axis measured from the centre of the base; for the base may be regarded as a polygon having an indefinite number of sides.

Obs. The observation made upon Art. 46, may be extended to the preceding proposition. That is to say, the centre of gravity of a tetrahedron is the same as that of four equal particles placed at its angular points.

48. *Given the centre of gravity of a heavy body, and also that of a certain portion of it, to find the centre of gravity of the remainder.*




Let $G$ be the centre of gravity of the body, $W$ its weight: $G_1$ the centre of gravity of the given portion, $W_1$ its weight. Join $G_1G$, and in that line produced take $G_2$, such that

$$G_2G : G_1G :: W_1 : W - W_1.$$

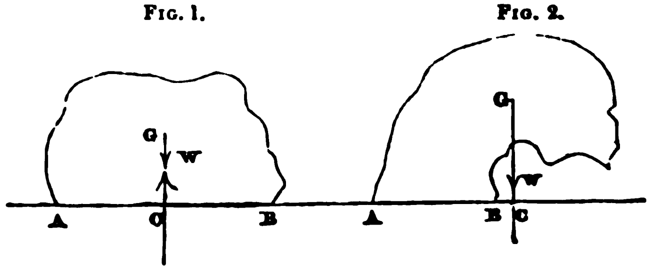
Then $G_2$ will be the centre of gravity required.

The preceding proposition is applicable to a variety of examples: it will enable us for instance to find the centre of gravity of the frustum of a pyramid or of a cone, that is, the centre of gravity of a portion of the body cut off by a plane parallel to the base.

The following is one of the most important properties of the centre of gravity.

49. *When a body is placed upon a horizontal plane, it will stand or fall according as the vertical line through the centre of gravity falls within or without the base.*

Fig. 1. Fig. 2.



Suppose the vertical line $GC$ through the centre of gravity $G$, to fall within the base, as in fig. 1; then we may suppose the whole weight of the body to be a vertical pressure $W$ acting in the line $GC$; this will be met by an equal and opposite pressure $W$ from the plane on which the body is placed, and so equilibrium will be produced and the body will stand.

But suppose, as in fig. 2, that the line $GC$ falls without the base; then there is no pressure equal and opposite to $W$ at $C$, and therefore $W$ will produce a moment about $B$, (the nearest point in the base to $C$,) which will make the body twist about that point and fall.

50. According to the proposition just proved, a body ought to rest without falling upon a single point, provided that it is so placed that the centre of gravity is in the vertical line passing through the point which forms the base. And in fact a body so situated would be, mathematically speaking, in a position of equilibrium, though practically the equilibrium will not subsist; this kind of equilibrium and that which is practically possible are distinguished by the names of *unstable* and *stable*. Thus an egg will rest upon its side in a position of *stable* equilibrium, but the position of equilibrium corresponding to the vertical position of its axis is *unstable*. The distinction between *stable* and *unstable* equilibrium may be enunciated generally thus: Suppose a body or a system of particles to be in equilibrium under the action of any forces; let the system be arbitrarily displaced very slightly from the position of equilibrium, then if the forces be such that they tend to bring the system back to its position of equilibrium the position is *stable*, but if they tend to move the system still further from the position of equilibrium it is *unstable*.

51. *When a heavy body is suspended from a point about which it can turn freely, it will rest with its centre of gravity in the vertical line passing through the point of suspension.*

For let $O$ be the point of suspension, $G$ the centre of gravity, and suppose that $G$ is not in the vertical line through

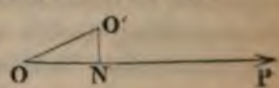
$O$; draw $OP$ perpendicular to the vertical through $G$, that is, to the direction in which the weight $W$ of the body acts. Then the force $W$ will produce a moment $W.OP$ about $O$ as a fulcrum, and there being nothing to counteract the effect of this moment equilibrium cannot subsist.


Hence $G$ must be in the vertical line through $O$, in which case the weight $W$ produces only a pressure on the point $O$, which is supposed immoveable.

## ON THE PRINCIPLE OF VIRTUAL VELOCITIES.

52. Def. If we suppose a point at which any force acts to be very slightly displaced, and from the new position of the point a perpendicular to be dropped upon the direction of the force, then the line intercepted between the foot of this perpendicular and the original position of the point is called the *Virtual Velocity* of the point of application of the force, or sometimes more briefly the virtual velocity of the force.

Thus let $O$ be the point at which the force $P$ acts, and suppose it to be slightly displaced so as to be brought into the position $O'$; from $O'$ draw the perpendicular $O'N$ on $OP$, then $ON$ is the Virtual Velocity of $P$.




If the displacement of $O$ is such that $N$ falls between $O$ and $P$, that is, if the virtual velocity is in the direction of the force, it is reckoned positive; if in the opposite direction, or $N$ on the other side of $O$, it is reckoned negative.

53. It will appear from what has been said that the virtual velocity of a force is to a considerable extent an arbitrary quantity; and such is the fact, but it will be observed that when we have several forces acting at different points of a rigid body the displacement of one point will in general determine the displacements of the others. For example, suppose we have two forces acting on the arm of a lever, then if we raise

one extremity of the lever through a small space, the other extremity is necessarily depressed through a space, the magnitude of which can be assigned.

54. If the displacement is made in the direction of the force, the whole displacement becomes, according to our definition, the virtual velocity, and if in a direction perpendicular to that of the force, the virtual velocity is zero. And in general we may regard the virtual velocity as the space through which the point of application is moved *in the direction of the force.* It will be seen also, by reference to Art. 56, page 154, that when a force is acting perpendicularly to the arm of a lever, and the arm is made to turn through a very small angle, the small arc of a circle described by the point of application may be taken as the virtual velocity of the force.

55. Hence we shall see something of the meaning of the term Virtual Velocity; for suppose we have any number of forces acting at different points, and that in consequence of an arbitrary motion of one of the points in the direction of the corresponding force through a very small space $a$, the other points of the system move in the direction of their respective forces through the spaces $\beta$, $\gamma$, &c.; then since these points move contemporaneously through the spaces $a, \beta, \gamma \ldots$, these spaces measure the *rate* at which they respectively move; for example, suppose $\beta = 2a$, $\gamma = 3a$, &c., then the points must have moved at rates, or with velocities, which are in the ratio of 1, 2, 3, &c.; but these velocities are not real, since the parts of the system do not move in consequence of the forces which act upon them; if they did move, the question would be Dynamical, not Statical; hence the small spaces of which we have been speaking are called *Virtual* Velocities. And the student cannot too carefully bear in mind, that the motion which would seem to be implied by the term velocity is altogether of a geometrical character, that is, it is not due to the forces of the system, but is only a displacement supposed to be arbitrarily produced without any reference to the nature of the forces necessary to produce it.

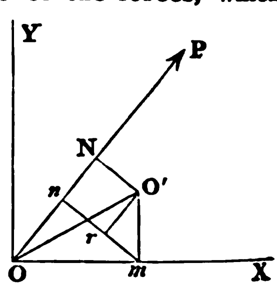
56. Having explained what is meant by virtual velocity, we shall be able to prove several propositions, which form particular cases of a very general principle known as that of Virtual Velocities, the proof of which we cannot give here, but of which it may be well to give the enunciation.

*When a system of bodies is in equilibrium under the action of any forces, then if the system be very slightly displaced, the sum of the products of the forces and their respective virtual velocities will equal zero.*

All that we shall do will be to prove this principle in those cases of equilibrium, which have been already considered, assuming the results which have been obtained.

57. *To prove the principle of virtual velocities in the case of a single particle, acted upon by any system of forces in the same plane.*

Let $O$ be the particle, $P$ any one of the forces, which makes an angle $\theta$ with a line $OX$ drawn through $O$. Let the particle be displaced to $O'$, and from $O'$ draw $O'N$ perpendicular to $OP$, and let $ON = p$; also draw $O'm$ perpendicular to $OX$, and let $Om = x$, $O'm = y$; then it is easy to see, by drawing $mn$ perpendicular to $OP$, and $O'r$ perpendicular to $mn$, that




$$p = On + O'r = x \cos \theta + y \sin \theta.$$

Similarly, if $p'p'' \dots$ be the virtual velocities of forces $P'P'' \dots$ acting at angles $\theta', \theta'' \dots$ with the line $OX$, we shall have

$$p' = x \cos \theta' + y \sin \theta',$$

$$p'' = x \cos \theta'' + y \sin \theta'',$$

&c. = &c.

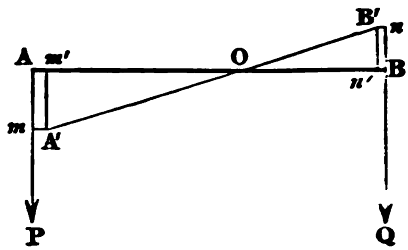
$$\therefore Pp + P'p' + P''p'' + \dots = x\,(P \cos \theta + P' \cos \theta' + P'' \cos \theta'' + \dots)$$
$$+ y\,(P \sin \theta + P' \sin \theta' + P'' \sin \theta'' + \dots) = 0,$$

by the general conditions of equilibrium established in Art. 15,

page 218, which proves the principle of virtual velocities in this case.

58. *To prove the principle of virtual velocities in the case of the Lever.*

(1) Suppose the lever to be a straight lever $AB$, having arms $AO = a$, $BO = b$, and to be acted upon by forces $P$ and $Q$ perpendicular to the arms.




Let the lever be turned through a small angle about its fulcrum, so that the points $A$, $B$, are brought into the positions $A'$, $B'$, respectively; from $A'$, $B'$, draw $A'm$, $B'n$ perpendicular to the directions of the forces, and $A'm'$, $B'n'$ perpendiculars upon the lever. Then $Am$, $Bn$, or $A'm'$, or $B'n'$ are the virtual velocities of $P$ and $Q$.

Now we have seen, Art 17, page 219, that

$$P \, . \, a = Q \, . \, b;$$

but by similar triangles $A'Om'$, $B'On'$,

$$\frac{A'm'}{a} = \frac{B'n'}{b};$$

$$\therefore P \, . \, A'm' = Q \, . \, B'n'.$$

Hence, not having regard to sign, we may say that

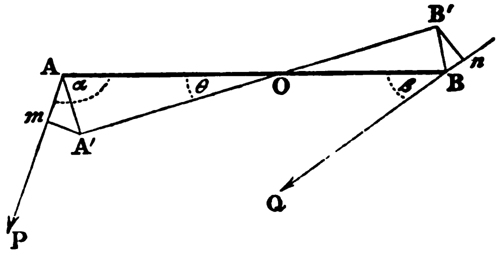
$$P \times P\text{'s virtual velocity} = Q \times Q\text{'s virtual velocity}.$$

Or if we denote $A'm'$ by $p$, and $B'n'$ by $-q$, (see Art. 52) we shall have

$$P \, . \, p + Q \, . \, q = 0,$$

which coincides with the general enunciation of the principle of virtual velocities given in Art. 56.

(2) Suppose the forces $P$ and $Q$ to act at any angles $a$ and $\beta$ with the lever $AOB$.




Let the lever be turned through a small angle as before, and call the angle $\theta$. From $A'$, $B'$ the new positions of $A$ and $B$ draw $A'm$, $B'n$ perpendicular to the directions of the forces; then if $\theta$ be indefinitely small, $Am$, $Bn$ will be the virtual velocities of $P$ and $Q$. Join $AA'$, $BB'$.

Then $Am = AA' \cos A'Am$

$$= 2a \sin\frac{\theta}{2} \cos\left(a - 90^\circ + \frac{\theta}{2}\right), \quad \left(\text{since } A'AO = 90^\circ - \frac{\theta}{2}\right),$$

$$= 2a \sin\frac{\theta}{2} \sin\left(a + \frac{\theta}{2}\right);$$

similarly it will be found that $Bn = 2b \sin\frac{\theta}{2} \sin\left(\beta - \frac{\theta}{2}\right)$,

$$\therefore \frac{Am}{Bn} = \frac{a \sin\left(a + \frac{\theta}{2}\right)}{b \sin\left(\beta - \frac{\theta}{2}\right)}.$$

If we make $\theta$ indefinitely small, we shall have $\sin\left(a + \frac{\theta}{2}\right)$ indefinitely nearly equal to $\sin a$, and $\sin\left(\beta - \frac{\theta}{2}\right)$ to $\sin\beta$,

$$\text{and therefore } \frac{P\text{'s virtual velocity}}{Q\text{'s} \ldots\ldots\ldots\ldots\ldots\ldots} = \frac{a \sin a}{b \sin\beta}.$$

But we know, from Art. 17, page 219, that

$$\frac{P}{Q} = \frac{b \sin\beta}{a \sin a},$$

$\therefore P \times P$'s virtual velocity $= Q \times Q$'s virtual velocity,

or, having regard to the signs of the virtual velocities, and calling them $p$ and $-q$,

$$P \cdot p + Q \cdot q = 0,$$

as before.

This last demonstration is applicable to the case of any rigid body acted upon by two forces in the same plane, and having one point fixed; for through the fixed point we may draw a straight line intersecting the directions of the forces, and the points of intersection we may regard as the points of application of the forces. Hence in this general case the principle of virtual velocities is true.

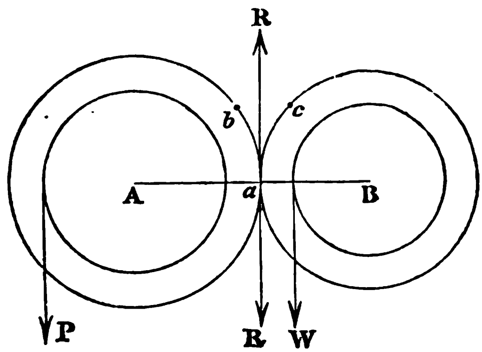
59. *The Wheel and Axle.*

The condition of equilibrium being precisely the same as for the straight lever acted upon by two forces perpendicular to its arms, the demonstration will be the same as in that case.

60. *Toothed Wheels.*

To simplify the investigation we shall suppose the teeth to be indefinitely small, and therefore the wheels themselves to be in contact; the action between the wheels will then be in the direction of the tangent to the wheels at the point of contact, or perpendicular to the line joining their centres.

Let $R$ be the action between the wheels, and suppose one of them to be turned through a very small angle, and let $b$, $c$



be the positions into which the points which were in contact at $a$ are brought by the displacement.

Then from what has been already proved, and from Art. 54, we shall have,

$$P \times P\text{'s virtual velocity} = R \times ab,$$

similarly,

$$W \times W\text{'s virtual velocity} = R \times ac;$$

but it is manifest that $ab = ac$, since those portions of the wheels have been in contact,

$$\therefore P \times P\text{'s virtual velocity} = W \times W\text{'s virtual velocity}.$$

61. *The Pully.*

In applying the principle of virtual velocities to pullies, we suppose the weight $W$ to be raised through a small space, which small space will be its virtual velocity, and the corresponding space through which the point of application of $P$ must be moved in order to keep the string stretched will be the virtual velocity of $P$.

(1) *The single moveable Pully.*

If in the figure, page 233, Art. 26, we suppose $W$ raised through a small space $a$, the string on either side of the pully will be shortened by the same quantity; consequently the point of application of $P$ must be raised through $2a$, which will be $P$'s virtual velocity.

But $$2P = W;$$

$$\therefore P \times 2a = W \times a,$$

or $P \times P$'s virtual velocity $= W \times W$'s virtual velocity.

(2) *The first system of Pullies.*

In the figure of page 234, Art. 27, let $W$ be raised through a small space $a$, then the lowest pully rises through a space $a$, the second (reckoning from the lowest) through a space $2a$, the third through $2 \times 2a$ or $2^2a$, and so on; hence the $n^{th}$

pully will rise through a space $2^{n-1}a$, and the space through which $P$ will descend will be $2^n a$.

But $$P \times 2^n = W;$$

$$\therefore P \times 2^n a = W \times a,$$

or $P \times P$'s virtual velocity $= W \times W$'s virtual velocity.

(3) *The second system of Pullies.*

In the figure of page 234, Art. 28, let $W$ be raised through a small space $a$; then if there be $n$ strings between the two blocks, each of these will be shortened by a quantity $a$, consequently $P$ will descend through a space $na$.

But $$P \times n = W;$$

$$\therefore P \times na = W \times a,$$

or $P \times P$'s virtual velocity $= W \times W$'s virtual velocity.

(4) *The third system of Pullies.*

In the figure of page 235, Art. 29, let $W$ be raised through a small space $a$; then the second pully (reckoning from the highest) will descend through a space $a$, and therefore the third pully will descend through $2a$; but in consequence of the rising of $W$, the third pully would have descended through $a$, even if the second had been fixed, therefore on the whole it descends through $2a + a$. In like manner the fourth descends through $2(2a + a) + a$ or $(2^2 + 2 + 1)\,a$; and the $n^{\text{th}}$ through $(2^{n-2} + 2^{n-3} + \ldots + 1)\,a$, and $P$ through

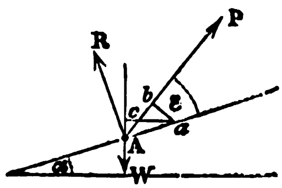
$$(2^{n-1} + 2^{n-2} + \ldots + 1)\,a, \text{ or through } (2^n - 1)\,a.$$

But $$P \times (2^n - 1) = W;$$

$$\therefore P \times (2^n - 1)\,a = W \times a,$$

or $P \times P$'s virtual velocity $= W \times W$'s virtual velocity.

62. *The inclined Plane.*



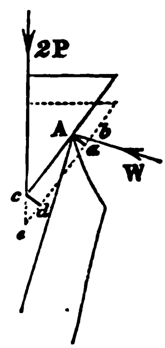

Let $A$ be a particle, of weight $W$, which is kept at rest on an inclined plane by a force $P$, the direction of which makes an angle $\epsilon$ with the plane; $R$ the pressure of the plane on $A$; $\alpha$ the angle of the plane.

Suppose $A$ to be moved along the plane to the point $a$; from $a$ draw $ab$, $ac$ perpendicular to the directions of $P$ and $W$ respectively; then $Ab$, $Ac$ are the virtual volocities of $P$ and $W$; $R$ will have no virtual velocity, Art. 54.

$$\text{Now} \qquad Ab = Aa \,.\, \cos \epsilon,$$
$$\text{and } Ac = Aa \,.\, \sin \alpha;$$
$$\text{but } P \cos \epsilon = W \sin \alpha;$$
$$\therefore P \times Aa \cos \epsilon = W \times Aa \sin \alpha,$$
$$\text{or } P \times Ab = W \times Ac,$$

or $P \times P$'s virtual velocity $= W \times W$'s virtual velocity.

63. *The Wedge.*




Let the figure represent one side of an isosceles wedge, acted upon by a force $2P$; $A$ the point of contact of the obstacle; $W$ the resistance; $\alpha$ the semiangle of the wedge.

Suppose the wedge depressed through a small space $ce$, so that the wedge assumes the position represented by the dotted line; and let $b$ be the corresponding point of contact of the obstacle, so that the point $A$ has moved through the very small space $Ab$, which we shall consider to be a straight line. From $c$, $A$, draw $cd$, $Aa$, perpendicular to the side of the wedge, and let the angle $bAa = i$.

$$\text{Then} \qquad Ab \cos i = Aa = cd = ce \sin \alpha;$$
$$\text{but} \qquad P \cos i = W \sin \alpha;$$
$$\therefore P \times ce = W \times Ab,$$

or $P \times P$'s virtual velocity $= W \times W$'s virtual velocity.

64. *The Screw.*

It is evident that if the arm upon which the force $P$ acts, (see fig. page 242), be made to describe a complete revolution, the weight $W$ will be raised or depressed through a space equal to the vertical distance between two threads of the

screw; and the same proportion will be observed whatever be the actual magnitudes of the motions of $P$ and $W$; consequently supposing these motions to be indefinitely small, we have (Art. 54)

$$\frac{P\text{'s virtual velocity}}{W\text{'s} \ldots\ldots\ldots\ldots\ldots} = \frac{\text{circumference of circle described by } P}{\text{vertical distance between two threads}};$$

$$\text{but } \frac{P}{W} = \frac{\text{vertical distance between two threads}}{\text{circumference of circle described by } P};$$

$$\therefore P \times P\text{'s virtual velocity} = W \times W\text{'s virtual velocity}.$$

65. We have thus proved the principle of virtual velocities in the case of all the simple machines. In any combination of these machines it is not difficult to conclude that the principle must also hold. A law which thus brings under one view the conditions of equilibrium in so many different cases will doubtless appear to the student one of great beauty and generality, although only a deduction from conditions previously established; but the principle of virtual velocities appears in its most striking light, when demonstrated in all its generality, and made the basis of mechanical investigations.

# DYNAMICS.

1. We have now to treat of force, considered as producing *motion* in bodies. Our first business will be to explain accurately what we mean by the *velocity* of a body, and how it is measured.

2. The *velocity* of a body is the *rate* of its motion, or the degree of quickness with which it is moving: if of two bodies one passes in a given time over twice the distance passed over by the other, we say that the velocity of the first is twice as great as the velocity of the second.

Velocity may be *uniform* or *variable*. By saying that a body moves with uniform velocity, we mean that it moves through equal spaces in equal times; when the velocity is variable this is not the case.

*Velocity when uniform is measured by the space passed over in a unit of time; when variable it is measured at any instant, by the space which would be passed over in a unit of time, if the body were to move during that unit with the velocity which it has at the proposed instant.*

This requires some explanation. Let us first consider *uniform* velocity; in order to measure it, we first fix upon some unit of time, that is, some convenient period of time to which we may always refer, and by which we may measure other periods: the unit agreed upon is one *second*, so that in what follows, (unless the contrary be stated,) time will be measured by *seconds*; if any number, as 5 for example, should occur as representing time, it will be understood to mean 5" or 5 *seconds*. We may here also state that, in like manner, we find it convenient to agree upon a fixed unit of space, and that the unit we shall take will be one *foot*, so that (to take our former example) the number 5, if representing

space, will mean 5·*feet**. With these conventions our rule for measuring uniform velocity will be, that it is measured by the number of *feet* described in one *second*; and it does not require much consideration to perceive, that this is a proper mode of measuring velocity; for suppose two bodies are moving uniformly, and that one passes over 3 feet in a second and the other 5 feet, then the numbers 3 and 5 are proper representatives of the respective velocities of the bodies. Any other numbers in the same proportion would be equally proper expressions for the velocities, and the actual numbers of course depend upon the particular units we have chosen; thus, if in the case just supposed we had taken 2 seconds as the unit of time instead of 1 second, the values of the velocities would have been 6 and 10 instead of 3 and 5.

With regard to the mode of measuring *variable* velocity, it will be seen, that when the rate of a body's motion is changing from moment to moment, we cannot measure its velocity by the space which it passes over in a unit of time, because it will not have moved at the same rate during the whole of that unit of time. Hence we adopt the method already enunciated, and we measure the velocity of a body at any moment, not by any space actually described, but by the space which *would be* described in one second, if the body moved during that time with the velocity with which it is animated at the moment in question. In doing this, we are in fact only adopting a method which is in ordinary use; for when we speak of the velocity of a coach as 10 miles per hour, we do not mean to assert, that the coach will pass over, or has passed over, 10 miles in any given hour, but only this, that *if* it were to proceed during an hour at the rate at which it was moving at the moment of our observation it *would* pass over 10 miles.

3. We shall in what follows usually denote time by the letter $t$, space by $s$, velocity by $v$. From what has been said, we shall be able to attach a distinct conception to each of these symbols; $t$ will be the number of *seconds* in any

* A *second* and a *foot* are here treated as defined portions of time and space respectively; the actual method of defining them need not here be considered.

time symbolized, $s$ the number of *feet* in any space, and $v$ the number of *feet* which are, or would be, described by a body in one *second*, according as the velocity is uniform or variable.

4. PROP. *If* s *be the space which a body, moving uniformly, with a velocity* v, *describes in the time* t, *then* $s = vt$.

For $v$ is the number of feet which the body passes over in one second; and the body passes over equal spaces in equal times, therefore in $t$ seconds it describes $vt$ feet, i.e. $s = vt$.

5. We may extend to velocity the convention respecting algebraical signs, which we have already found of use in the case of lines, angles, and statical forces. If we fix upon any point in a body's path, and consider the velocity of the body to be positive when its distance from that point is increasing, then we must regard the velocity as negative when that distance is diminishing: for instance, suppose a body is projected upwards from the earth's surface, and we regard the velocity during the ascent as positive, then when the body descends the velocity will be negative.

6. It is manifest that we may represent velocity, in the same way as we formerly represented force, by a straight line, the length of the line indicating the magnitude of the velocity, and the direction in which it is drawn that in which the body is moving; this we shall call representing a velocity in direction and magnitude.

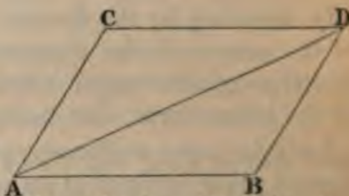
7. A body can have at any instant of its motion only one determinate velocity; that is to say, it must be at every instant moving at a certain rate and in a certain direction; nevertheless we may regard the velocity of the body as the *resultant* of two or more *component* velocities, if we understand by this phraseology that the component velocities are such, that if they were simultaneously impressed upon the body the motion of the body would be that which it actually is. Suppose, for instance, that a body moves uniformly down an inclined plane; then

the body has a velocity along the plane and in no other direction; we may however speak with convenience and propriety of the body having a vertical velocity and a horizontal velocity, meaning by the former of these terms the rate of increase of the perpendicular distance of the body from the horizontal line through the upper extremity of the plane, and similarly of the latter term. Let us take a particular example: suppose a body to move uniformly, at the rate of $\sqrt{2}$ feet in a second, down a plane inclined at an angle of 45° to the horizon: then at the end of the first second the body will be distant $\sqrt{2}$ feet from the point of starting, and it will consequently be 1 foot distant from the horizontal line through that point: at the end of two seconds it will be distant $2\sqrt{2}$ feet from the starting point, and 2 feet from the horizontal line; and so on; hence we may speak of the body having a *vertical velocity* of 1 foot per second; and in like manner it will have a *horizontal velocity* of 1 foot. Thus we *resolve* the velocity $\sqrt{2}$ in the direction of a line inclined at 45° to the horizon, into a vertical and a horizontal velocity each equal to 1; and conversely, we may speak of the former velocity resulting from the *composition* of the two latter. And hence we are led to a system of composition and resolution of velocities analogous to the composition and resolution of forces treated of in Statics; only it is to be observed, that the composition and resolution of velocities involves no considerations of the nature and properties of matter, but is purely geometrical.

We shall be able to express the general rule for the resolution and composition of velocities by a proposition analogous in form to the *Parallelogram of forces;* the method of representing velocities by straight lines will in fact enable us to enunciate the rule in the form of a proposition which may be called the *parallelogram of velocities*, a proposition which after what has been said will perhaps scarcely require proof, but which for distinctness' sake we shall enunciate and prove as follows.

8. PROP. *If two velocities, with which a particle is simultaneously animated, be represented in direction and magnitude by two straight lines drawn from the particle, the resultant*

*velocity of the particle will be represented in direction and magnitude by the diagonal of the parallelogram described upon those two straight lines.*



Let $A$ be the particle, $AB$ $AC$ the lines representing the two component velocities, $AD$ the diagonal of the parallelogram $ABDC$ described upon them.

Then, under the influence of the velocity $AB$ only, the particle would at the end of one second be at $B$, and under the influence of $AC$ it would be at $C$; suppose now, that instead of moving for one second under the influence of the two velocities, the particle moves for one second under the influence of $AB$, and then for one second under the influence of $AC$, its place at the end of the time will be the same as upon the former hypothesis. But on this supposition the particle will be at the end of the first second at $B$, and (drawing $BD$ parallel and equal to $AC$) it will be at the end of the next at $D$; therefore the supposition of the particle being animated simultaneously with a velocity represented by $AB$, and $AC$ or $BD$, leads to the same result as to suppose the particle animated with the velocity represented by $AD$, since the motion at the end of one second is the same upon the two suppositions: in other words, $AD$ represents the resultant velocity in direction and magnitude.

Conversely, the velocity $AD$ may be resolved into the two velocities $AB$, $AC$; or a particle, whose actual velocity is represented by $AD$, may be conceived of as being animated by two simultaneous velocities $AB$, $AC$. A velocity may of course be resolved into two other velocities in an indefinite number of different ways; or still more generally we may resolve it into any number of component velocities, and we shall thus have a *polygon of velocities* as we had a *polygon of forces;* in fact if we take any polygon, (the sides of which may lie in the same plane or not,) and if one side represent the velocity of a particle in magnitude and direction, then this velocity may be regarded as the resultant of all the velocities represented by the sides of the polygon taken in order.

OBS. In this proof we have, for simplicity's sake, spoken of the velocities as if they were uniform; the same proof however applies, with a change of phraseology, to variable velocities.

9. It follows from the proposition just proved, that, if a particle be moving with a velocity $v$ in a direction making an angle $\theta$ with a given line, we may conceive the particle to be animated by two velocities $v \cos \theta$ in the direction parallel to the given line, and $v \sin \theta$ in the direction perpendicular to it. This is called *resolving* a velocity.

And hence we may compound any number of velocities and find their resultant, by a process exactly similar to that made use of for forces, and given at length in Art. 14, page 216.

In order further to elucidate this subject, let us deduce the rule for resolving velocities at once from the definition of resolution and composition of velocity, without the medium of the preceding proposition.

Let a particle move uniformly, with velocity $v$, along a plane inclined to the horizon at an angle $\theta$. Then at the end of one second its distance from the starting point is $v$, its distance from the horizontal line through that point is $v \sin \theta$, and its distance from the vertical line through the same $v \cos \theta$; at the end of two seconds these three distances become respectively $2v$, $2v \sin \theta$, and $2v \cos \theta$; and so on. Hence the particle may be said to be moving vertically with a velocity $v \sin \theta$, and horizontally with a velocity $v \cos \theta$; in other words the velocity $v$ may be *resolved* into the two velocities, $v \sin \theta$ vertical, and $v \cos \theta$ horizontal.

10. Having said thus much respecting velocity, we shall now proceed to treat of *force*, considered dynamically*. We have already defined force (page 203) as *any cause which changes, or tends to change, a body's state of rest or motion*, and we have already in the treatise on Statics considered the case of force *tending* to move bodies from rest; we shall now be employed with the case of force actually producing or changing motion.

* Here commences the subject of *Dynamics* properly so called; the preceding articles on velocity belong to what is sometimes distinguished as *Kinematics*.

We shall begin by enunciating the following general property of matter, which is known as the *First Law of Motion.*

*A body under the action of no external force will remain at rest, or move uniformly in a straight line.*

With regard to the *meaning* of this law, it is intended to assert that there is in matter no tendency to motion of one kind more than another, or indeed of any kind; that matter is purely inert*, and the cause of any motion which a body may have is to be sought not in the properties of the body itself, but in external influences. The property which matter possesses, of moving only with the velocity it has acquired from the action of external force, is called its *inertia;* hence we may say briefly, that the first law of motion asserts the *inertia* of matter.

11. As to the *proof* of this law, we may obtain some hint of its truth by observing, that the more nearly we make the circumstances of a body agree with those supposed, the more nearly is the law verified: for instance, according to the law the destruction of the velocity of a body moving along a dead level ought to be wholly due to friction, and we do in fact find that the more we guard against friction the longer the body will continue in motion; and on a railway, where the friction is very much diminished, the distance to which a train will proceed after the steam has been turned off is very great indeed. Other examples will suggest themselves to the student, or perhaps he may think the law so simple as not to require illustration; he must remember however that the want of a clear perception of its truth was for a long time a bar to progress in dynamical science, because men, misled by terrestrial

* When however we speak of matter as inert, it must be understood that we are speaking of each particle of matter so far as regards itself; in no other sense is matter inert, for each particle influences every other, and the motion of any one particle is due to the united action of all other surrounding particles. This general property of matter, the discovery of which is due to Newton, and the laws and consequences of which were investigated by him, and have been extended by other mathematicians, is known under the name of *universal gravitation.*

phænomena, considered it necessary to inquire what force was required to keep a body in a certain uniform state of motion, instead of inquiring what force was required in order to account for observed deviations from uniform rectilinear motion. A satisfactory proof of the truth of this, as well as of other laws which we shall meet with hereafter, arises from the accurate agreement with fact of calculations, many and complicated, which are based upon it. Possibly the mind, which has dwelt long on the subject, will see the truth of the law as necessarily involved in the idea of matter, and as having therefore an axiomatic character more convincing than any proof founded upon the agreement of calculation and experiment: we must not however pursue a remark, which would lead us into discussion unsuited to the character of the present elementary work.

12. Dismissing then the question of the nature of the proof of the first law of motion, and supposing its truth to be established, we learn from it that whenever a body is moving with a velocity varying either in direction or in magnitude, we are to conclude that the body is acted upon by some extraneous force. Confining ourselves, for distinctness of conception, to the case of a body moving in a straight line but not uniformly, we are to conclude that the change of velocity from one moment to another is due to the action of some force upon the body; hence the rate of this change must in some way be an index of the intensity of the force: but here the question occurs,—Are we to estimate the force acting on the body solely by the change of velocity, or are we to take into account also the quantity of matter contained in the body, for it is an obvious fact that a given force (as a blow for instance) will generate a greater velocity in a body as its weight is less? The answer is, that we may estimate the force either way, that is, we may estimate it either by the velocity generated only, or we may take into account not only the velocity generated, but also the quantity of matter moved; in the former case we call the force the *accelerating* force acting on the body, in the latter the *moving* force.

As the distinction between these two ways of estimating force is of first importance to a clear understanding of dynamical principles, we shall enunciate it as plainly as we can.

13. *Force considered with reference to velocity generated only, and not to the quantity of matter moved, is called* accelerating *force.*

*Force considered with reference to the quantity of matter moved, as well as the velocity generated, is called* moving *force.*

We trust from the manner in which we have introduced the two terms *accelerating* and *moving* force, that the student will not be misled into the notion that we are treating of two different kinds of force under those names. It may be well however to guard him against that danger, and to remind him that when we speak of accelerating force we speak of force regarded in a particular way, namely, as measured by the acceleration produced. It may be admitted that the choice of the names is not altogether faultless, and that it would perhaps be better instead of speaking of accelerating force to speak of the accelerating effect, or the velocity-measure, of a force; the terms are however current, and are not inconvenient if the precise meaning of them be borne in mind.

14. The exact relation between accelerating and moving force will be for our consideration presently: the former will just now occupy our attention, and we must consider carefully the mode of measuring it. We will observe, by the way, that the term *accelerating* is not used as opposed to, but rather as inclusive of, *retarding* force, the latter being supposed to differ from the former merely in its algebraical sign; and hence, when we speak of a force *generating* a velocity in a body, we intend the term to include forces which *destroy* velocity.

Accelerating force may be *uniform* or *variable*; it is called *uniform* when equal velocities are generated in equal times, *variable* when this is not the case.

*Accelerating force is measured, if uniform, by the velocity generated in a unit of time; if variable, by the velocity which*

*would be generated in a unit of time if the force remained constant during that unit.*

Let us illustrate the mode of measuring accelerating force, by reference to the case of the earth's attraction. We observe that bodies fall with a variable velocity, hence we know that they are acted upon by some force; this force we are led to attribute to an attraction residing in the earth, and which we call *gravity*. Again, we observe, (or at least we will suppose it to be observed,) that the increments of velocity of a falling body in equal times are equal; hence we conclude that gravity is a *uniform* force. Let us consider how it is to be measured; a body in one second is found to fall from rest through 16.1 feet, at the end of one second it is found to be moving with such a velocity that it would, if it continued to move with the velocity which then animates it, pass over in the next second 32.2 feet. Hence 32.2 feet is the measure of the velocity which has been generated in one second, and is therefore the measure of the accelerating force of gravity. It may be mentioned here, that this quantity is usually denoted by the letter $g$.

It appears then that 32.2 feet is the measure of the earth's attraction, and in making use of this result the student is requested to bear carefully in mind all the conventions upon which it depends. We have assumed that the accelerating force of the earth's attraction on all bodies is the same; an experimental proof of this is supplied by the fact, that under the exhausted receiver of an air-pump all bodies fall equally rapidly; the difference of velocity of falling bodies in air is due entirely to the different action of the air upon them.

15. Prop. *If* v *be the velocity generated by a uniform accelerating force* f *in the time* t, *then will* v = ft.

For $f$ is the expression for the velocity generated in one second; and since the force is uniform an equal velocity is generated in each second; therefore the velocity generated in $t$ seconds is $ft$; i.e. $v = ft$.

Thus in the case of the earth's attraction, the velocity generated in one second in a falling body is 32.2 feet, therefore

the velocity generated in 2 seconds is 64.4 feet; by which we mean, that if a heavy body be let fall, it will at the end of 2 seconds be moving with such a velocity that, were that velocity continued constant through one second, the body would move in that second through 64.4 feet.

16. We have said that force is measured by the velocity generated in a unit of time, but now we must observe further that there is a certain kind of force which cannot be so estimated. This kind of force is what we term *impulsive*, under which name we include all forces which are of the nature of a sudden blow, applying the name of *finite* to all other forces, such as that arising from the gravitation of bodies. We will first give formal definitions of these two classes of force, and then attempt to explain clearly the difference between them, and the necessity for different modes of measuring their effects.

DEF. A *finite* force is one which requires a finite time to generate a finite velocity.

DEF. An *impulsive* force is one which generates a finite velocity in an indefinitely short time.

17. Let us consider what takes place, when an impulsive force results from the impact of two bodies upon each other. For distinctness of conception, let us suppose that the impact is that of two ivory balls: when the balls strike each other, they appear to fly asunder instantaneously, but in reality a rather complicated action takes place between them; the first effect after impact is a compression of each of the balls, very slight of course, but nevertheless necessarily existing unless the balls be absolutely rigid, which no substance in nature is; after the compression has ceased, a restitution of the figure of the balls commences, in consequence of their *elasticity*, and when the form of the balls is restored they separate, and their action on each other ceases. The whole action just described necessarily occupies a certain space of time and cannot be instantaneous, and there is nothing in the nature of the forces considered to make them differ essentially from those which

we have denominated *finite;* nevertheless, if we attempt to measure them in the same way we are met by this insuperable difficulty, that we know nothing of the laws according to which the action we have described takes place; we cannot observe the action of the forces, because for all purposes of observation that action is instantaneous; hence we distinguish these *impulsive* forces as a new class of forces, not because they are physically different from those which we call finite, but because, since they generate a finite velocity in a time so short as to be considered indefinitely small, we are compelled to measure them in a different way; and we measure the accelerating force of an impulse, not by the velocity which would be generated in a unit of time, but by the whole velocity actually generated.

It will be seen from what has been said, that no force in nature can entirely coincide with the definition which we have given of impulsive force. We may in fact regard that definition as a mathematical fiction, to which certain actual forces approximate sufficiently nearly to justify us in regarding them as of the nature of impulses; in like manner as we have in Statics treated of rigid bodies, although no bodies in nature are rigid according to the strict definition of the term.

The measure of the *moving force* of an impulse will be considered, when we come to speak in general of the connection between accelerating and moving force.

18. The first Law of Motion would enable us, with the help of such mathematical calculation as we shall hereafter employ, to solve all problems of rectilinear motion; but we have at present no means of determining the motion of a body which is moving under the action of various forces in different directions, or of a body which being in motion is acted upon by a force not in the direction of the motion. That which we require is furnished us by the *Second Law of Motion*, which we shall at once enunciate.

19. *When any number of forces act upon a body in motion, each produces its whole effect in altering the magnitude and*

*direction of the body's velocity, as if it acted singly on the body at rest.*

The application of this law is as follows: At any moment a body has a certain velocity, also if it were at rest each of the forces would, acting separately, generate in it a certain velocity in a certain direction; we must suppose that the body has all these velocities simultaneously, and if they be compounded by the rule of the parallelogram of velocities, the resultant velocity will be the actual velocity of the body.

20. With regard to the *proof* of the second Law of Motion, we may appeal to such experiments as the following. A ball let fall from the mast of a ship in motion, will fall at the foot of a mast, notwithstanding the onward course of the ship: in this instance, the velocity of the ball is compounded of that which it has in consequence of the motion of the ship, and of that impressed upon it by the action of gravity; for before the ball is let fall, it has the same horizontal velocity as the ship, and when let fall, it does not lose this velocity, but only has it compounded with another, viz. that due to gravity; and as the ship also retains the same horizontal velocity, (for it is supposed to be sailing uniformly,) the ball falls, with reference to the deck, exactly as if the vessel were at rest.

Illustrative of this law also are such facts as these; that we walk with the same ease in different directions along the earth's surface, although we know the earth to be not at rest; that we can walk across a railway carriage in rapid motion; and doubtless many others will suggest themselves to the mind of the thoughtful student. Perhaps the most satisfactory is this; the time of oscillation of a pendulum is independent of the plane in which it vibrates, notwithstanding the earth's motion.

Further experimental proof arises from the fact of calculations, based upon this law, leading in the most delicate cases to correct results. Perhaps however it may be said, that when the idea of force has been completely seized and thoughtfully considered, the truth of the second Law of Motion will present itself to the mind in a form rather

axiomatical than experimental; but this is a question on which we shall not here raise a discussion.

21. Hitherto we have considered force only as measured by the velocity which it generates, in other words, we have been concerned with *accelerating force:* we must now examine the proper mode of estimating force, when we take into account the quantity of matter moved, or when we regard it as *moving force.*

The quantity of matter in a body is usually termed its *mass;* but how are we to define the term mass? If we were to conceive matter to be made up of ultimate atoms, which atoms should be all precisely alike in magnitude and in all their qualities, then we might measure the mass of a body by the number of atoms it contains; but this is not a practicable method, and we are compelled to measure the mass of a body by its effects; we measure the mass of a body by its *weight,* or we consider two bodies to be of the same *mass* if they are of the same *weight.* In this mode of estimating mass, it is assumed that the attraction of the earth on all particles is the same, that is, that the attraction is not dependent upon the nature of the matter attracted; an assumption which appears to be justified by the fact already alluded to, that bodies of all kinds fall with equal velocity under the exhausted receiver of an air-pump, and of the truth of which Newton assured himself by a variety of experiments*. We can only define the mass of a body

* Newton commences the Principia with a definition of *mass;* he tells us that "the quantity of matter in a body is measured by the product of its density and magnitude." And he explains his definition thus: "Air when its density is doubled, and when it fills twice the space which it did before, must be quadrupled in quantity; the same is true of snow or dust condensed by compression. And the same holds of all bodies condensed, by whatever cause. This *quantity of matter* I denote by the name of *Body* or *Mass.* And the mass of every body is determined by its *weight;* for I have ascertained by most accurate experiments with pendulums that the mass is proportional to the weight."

Here it will be observed, that although Newton speaks of the mass being measured by the volume and the density, yet the ultimate standard of reference is the *weight,* that being in fact the only test of density.

As mentioned in the text, it is clearly assumed that the action of the earth's attraction is the same on all kinds of matter, and only varies in different bodies in consequence of their containing different quantities of matter. For illustration; a cubic inch of lead

then, as being a quantity proportional to its weight, so that, if $W$ be the weight of a body, and $M$ its mass,

$$W \propto M,$$

$$\text{or } W = C.M,$$

where $C$ is some constant quantity, the numerical value of which will depend upon circumstances to be presently explained.

The method of defining mass deserves further explanation. The method of defining it already given is what may be called the *statical* method; we might however have defined mass dynamically, as is sometimes done; we might say that two bodies were of equal mass when the same force would in the same time generate in them equal velocities: it could then be made matter of experiment whether the same force would in equal times generate equal velocities in bodies of equal weight, and if this should prove to be the case then we might substitute for the dynamical definition of mass the more convenient statical definition, namely, that bodies are of equal mass when they are of equal weights. We, on the other hand, have commenced by defining mass statically, and we shall have presently to connect this statical definition with the dynamical effects of force.

22. Before proceeding further, we must define the term *Momentum*, which we shall use frequently.

Def. The *momentum* of a particle of matter is its mass multiplied by its velocity; the momentum of a body, or system of particles, is the sum of the products of the masses of the particles by their respective velocities.

weighs more than a cubic inch of wood, and it might be argued that this was on account of the greater intensity of the earth's attraction for lead than for wood, even as in chemistry substances have a greater affinity for one substance than another; but this is exactly what is denied in the case of the earth's attraction, and it is asserted that the greater gravitation of the lead than of the wood is due to the greater degree of closeness with which the particles in it are packed. Without entering upon other experiments, the fact mentioned in the text, namely, the equal velocity of all substances under an exhausted receiver, is very strong in favour of the assertion; because, if there were any difference in the action of the earth on different substances, we might expect it to be shewn in the difference of velocity generated as well as in that of pressure produced.

The phrase *quantity of motion* is sometimes used instead of *momentum*.

23. Let us now inquire concerning the method of measuring *moving force:* in other words, when pressure communicates motion to a body, what will be the proper measure of its effect? The answer is given by the *Third Law of Motion*, which we may enunciate as follows:

*When pressure produces motion in a body, the momentum generated in a unit of time, supposing the pressure constant, or which would be generated supposing the pressure variable, is proportional to the pressure.*

24. The result of this law is, that as *velocity* generated is the measure of *accelerating* force, so *momentum* generated is the measure of *moving* force. And if $P$ be a uniform pressure, $v$ the velocity generated in the time $t$, in a body the mass of which is $M$, we shall have

$$Mv = Pt.$$

Suppose $f$ to be the accelerating force corresponding to the pressure $P$, then we have

$$v = ft,$$

hence, comparing the last two formulæ,

$$P = Mf,$$

or, moving force = mass × accelerating force.

Thus the third Law of Motion teaches us how, when a pressure is given, to deduce the accelerating force due to it, for we have only to divide the pressure by the mass moved and we have the accelerating force required. Let us take the example of the pressure caused by the weight of a body; let $W$ be the weight, $M$ the mass; then we know that the accelerating force is that which was before denoted by $g$, hence by the third Law of Motion we must have

$$W = Mg^*,$$

and therefore the constant $C$ in the formulæ of Art. 21, is to

* The adoption of this formula compels us to take a peculiar quantity as the unit of weight in Dynamics.

that he will derive assistance from examining the [illegible] treated in quite a different manner. We should [illegible]ecommend the omission of this article in the first [illegible] the subject; the student can scarcely fail to profit [illegible]ng the manner in which dynamical principles pre[illegible]mselves to the mind of [illegible] but he will profit if he has a tolerably good previous knowledge.

OF FORCE WHEN [illegible]

shewing how to take account of the masses as affecting the [illegible] when they act upon different bodies. [illegible] act expression, which is frequently employed, and which [illegible] thought.

[illegible]hat a body is placed upon a horizontal plane, and that it is [illegible]from moving by any friction. If I wish to make the body [illegible] plane, it will be necessary, on account of the inertia of [illegible] use of a certain effort: if to this body we join a [illegible] third, and so on, it will be necessary to employ, in order [illegible] same motion, a force more and more considerable. I [illegible]ach case, the sensation of having an effort to make; but I [illegible] from this that the matter offers any resistance to the [illegible] there exists in bodies what is very improperly called a *force of* [illegible] we express ourselves thus, we confound the sensation which we [illegible] which results from the effort which we make, with the [illegible]istance which in reality does not exist.

[illegible] plane is not perfectly smooth there is a resistance to hori[illegible] and I cannot displace the body upon the plane without [illegible]t which shall produce a force greater than this resistance. [illegible] if I wish to raise the body vertically, there is also a re[illegible] motion which I must overcome by a superior force. In [illegible]ll not produce any motion so long as I do not exert an [illegible]n the weight of the body, or than its adhesion to the [illegible] but if we suppose the existence of neither gravity nor [illegible]t the body in motion, however small an effort I exert, [illegible] may be the mass of the body: hence, if I find that it [illegible]ke a greater effort to communicate the same motion to [illegible]nother, I conclude that the first is composed of a greater [illegible] than the other; and if I could compare with precision [illegible] have exerted, their ratio would be the same as that of [illegible]dies. It is upon a similar principle that the measure [illegible]es is founded, that is, the principle of reference to the [illegible]ces which will put them in like states of motion, [illegible]hich is founded the method of measuring forces when [illegible]ass of the bodies moved and of the velocities generated.

be replaced by $g$, the accelerating force of the earth's attraction*.

If in the formula $P = Mf$, we make $M = 1$, we have $P = f$, in other words accelerating force may be regarded as the moving force upon a unit of mass; and this is a very useful

For let $\rho$ be the density of a body of which the mass is $M$ and the volume $V$; in other words, let $\rho$ be such a quantity that $M = \rho V$, then

$$W = \rho V g.$$

In this formula let $W = 1$, and $\rho = 1$,

$$\therefore V = \frac{1}{g},$$

that is, the unit of weight is the $g^{\text{th}}$ part of the weight of a unit of volume of the substance whose density is taken for unity. In general, distilled water at a given temperature is taken as the standard substance, and a cubic foot as the unit of volume, and the weight of a cubic foot of distilled water is found to be 1000oz: hence the unit of weight in Dynamics must be $\frac{10000}{322}$ oz.

* The Third Law of Motion is enunciated in a different form by Newton, as follows:

*Reaction is always equal and opposite to action; or the mutual actions of two bodies upon each other are always equal and in opposite directions.*

In illustration of the Law Newton has these remarks:

"Whatever body presses or draws another, is in like manner itself pressed or drawn. If a stone is pressed with the finger, the finger is also pressed by the stone. If a horse draws a burden, the horse will be (so to speak) also drawn back by the burden: for the rope by which the burden is drawn being equally stretched throughout will pull the horse towards the burden as much as the burden towards the horse, and will impede the motion of the one as much as it expedites the motion of the other. If one body impinging upon another change its motion in any manner, it will also itself undergo the same change in its own motion only in the opposite direction, on account of the equality of the mutual pressure. It is the change of motion, (*momentum*,) not of velocity, which is equal to the action in such cases, at least in bodies otherwise moving freely. For the change of the velocities of two impinging bodies, since the momenta of the two are equally changed, is inversely as the masses of the bodies."

From this it will be seen, that the third Law of Motion as enunciated by Newton is not entirely equivalent to the Law as stated in the text; the reason is that he has before, namely, in the second of the definitions which preface the Principia, laid down, that the quantity of a body's motion is to be measured by the product of the mass and the velocity; taken in conjunction with this definition the law is equivalent to that of the text; for Newton's third Law asserts that action and reaction are necessarily equal and opposite, and if the action be represented by a pressure and the reaction by the motion produced the equality must hold; it is not possible to conceive it otherwise. But how to measure the motion produced? Newton has before laid down the rule, that it is to be measured by that which we have called *momentum*; hence his assertion, that action and reaction are equal and opposite, becomes equivalent to that of the text, that the momentum generated by a pressure is proportional to the pressure.

mode of considering the subject, for it will tend to enforce the remark which was formerly made, namely, that what we call accelerating force and moving force are not different kinds of force, but force considered in two different ways: if we compare two forces by their effects upon different masses, then we must take account of those masses, or we must estimate the forces as *moving forces;* if on the other hand we select a particular mass and compare the forces by their effects upon that mass, then we estimate the forces as *accelerating* forces.

We may exhibit the third Law of Motion from a slightly different point of view, by saying, that it establishes a relation between the statical measure of force, which is the pressure produced, and the dynamical measure, which is the velocity generated. This remark perhaps requires elucidation; let us take the case of gravity, or the weight of bodies; in statical problems it is sufficient to estimate the weight of a body by the number of pounds to which it is equivalent, but this method is evidently not sufficient in Dynamics, inasmuch as it only gives us the ratio of one weight to another without affording any help towards measuring the absolute effect of the weight in producing motion. Suppose, however, that in Statics we agree upon speaking of bodies as of different mass according as their weights are different, that is, suppose that we agree upon considering the mass of a body $A$ as being twice as great as that of a body $B$ of the same volume, or the density of $A$ as being half that of $B$, when the weight of $A$ is twice as great as that of $B$; suppose also that at the same time in Dynamics we agree upon speaking of the mass of $A$ as being twice as great as that of $B$, when the same force will in the same time generate only half the velocity in $A$ which it will in $B$; then the question is, whether these two distinct methods of defining mass lead to the same results, and the third Law of Motion as we have enunciated it assures us that they do.

25. This Law of course includes *impulsive* forces, and we may therefore say that an impulsive force is measured by the whole momentum generated by it. (See Art. 17.)

26. With regard to the truth of the third Law of Motion, it may be considered either to rest on experiment, confirmed by the coincidence with fact of innumerable calculations founded upon it, or perhaps to be deducible without experiment from previously established principles.

We shall content ourselves with giving an account of one experiment, divested of all the refinements by means of which *Atwood* was able, with a machine constructed for the purpose, to obtain great accuracy of result*.


Let $W$ and $W'$ be two weights, of which $W$ is the greater, connected by a fine string passing over a small fixed pully. Then $W$ will descend, and as $W$ and $W'$ are pulling in opposite directions, the weight producing motion is $W - W'$, and the weight moved is $W + W'$. The advantage of this experiment is, that we can alter the difference between the weights $(W - W')$ as much as we please, while the weight moved $(W + W')$ remains the same. Now by numerous experiments it was determined, that the ratio $\dfrac{W - W'}{W + W'}$ is in all cases proportional to the accelerating force, or that the pressure producing motion is proportional to the product of the weight moved and the accelerating force, or (which is the same thing) the product of the mass and the accelerating force: which result is in accordance with the third Law of Motion.

27. The mode of estimating force, when the mass of the body moved is taken into account, being confessedly a difficult subject to grasp, we shall exhibit it from another point of view. The following articles are taken, with some omissions and adaptations, from the *Traité de Mécanique* of Poisson, and, as will be noticed, do not refer to the third Law of Motion; the treatment of the principles of Dynamics under the heads of the three Laws of Motion is in fact a purely English method, and is taken from the Principia of Newton; it has its advantages, at the same time the student will pro-

* *Atwood's* machine has lately been much simplified, and adapted to the purposes of a lecture-room, by Professor Willis.

bably find that he will derive assistance from examining the subject as treated in quite a different manner. We should however recommend the omission of this article in the first reading of the subject; the student can scarcely fail to profit by observing the manner in which dynamical principles presented themselves to the mind of Poisson, but he will profit much more if he has a tolerably good previous knowledge.

MEASURE OF FORCE WHEN THE MASS MOVED IS TAKEN INTO ACCOUNT.

1. "Before shewing how to take account of the masses as affecting the action of forces when they act upon different bodies, it is desirable to rectify an inexact expression, which is frequently employed, and which tends to confusion of thought.

"Conceive that a body is placed upon a horizontal plane, and that it is not prevented from moving by any friction. If I wish to make the body slide upon the plane, it will be necessary, on account of the inertia of matter, to make use of a certain effort; if to this body we join a second, then a third, and so on, it will be necessary to employ, in order to produce the same motion, a force more and more considerable. I shall have in each case, the sensation of having an effort to make; but I must not conclude from this that the matter offers any resistance to the effort, and that there exists in bodies what is very improperly called a *force of inertia*. When we express ourselves thus, we confound the sensation which we experience, and which results from the effort which we make, with the sensation of a resistance which in reality does not exist.

"When the plane is not perfectly smooth there is a resistance to horizontal motion, and I cannot displace the body upon the plane without exerting an effort which shall produce a force greater than this resistance. In like manner, if I wish to raise the body vertically, there is also a resistance to this motion which I must overcome by a superior force. In both cases, I shall not produce any motion so long as I do not exert an effort greater than the weight of the body, or than its adhesion to the horizontal plane; but if we suppose the existence of neither gravity nor friction, I shall put the body in motion, however small an effort I exert, and however great may be the mass of the body: hence, if I find that it is necessary to make a greater effort to communicate the same motion to one body than to another, I conclude that the first is composed of a greater quantity of matter than the other; and if I could compare with precision the efforts which I have exerted, their ratio would be the same as that of the masses of the bodies. It is upon a similar principle that the measure of the masses of bodies is founded, that is, the principle of reference to the magnitudes of the forces which will put them in like states of motion, and, conversely, upon which is founded the method of measuring forces when we take account of the mass of the bodies moved and of the velocities generated.

2. "Two material particles, belonging to bodies which are of different natures, have their masses equal or unequal, according as equal forces impressed upon them generate in the same time equal or unequal velocities. Suppose, for distinctness' sake, that the forces applied to these two particles are vertical, and that when placed in the opposite scales of a balance there is equilibrium. These forces will on this hypothesis be equal; and this being so, if the two particles are made entirely free, and the same forces put them in motion, their masses will be equal or unequal, according as the indefinitely small velocities which they receive in the first instant of their motion are equal or unequal.

"When, in this manner, the masses of different material particles have been ascertained to be equal, we can by joining them form other particles having their masses in any given ratio. Thus, if we call the mass of each of the equal particles $\mu$, and denote by $m$ and $m'$ the masses of two other particles formed respectively of $n$ and $n'$ of the first, we shall have

$$m = n\mu, \qquad m' = n'\mu.$$

"Now let $u$, $v$, $v'$, be indefinitely small velocities, $i$ and $i'$ whole numbers, and let

$$v = iu, \qquad v' = i'u.$$

"If two forces $f$ and $f'$ impress on the masses $m$ and $m'$ the velocities $v$ and $v'$ in the same instant of time, then will the following proportion be true,

$$f : f' :: mv : m'v'.$$

"For, we may in fact regard $f$ as the sum of a number ($n$) of equal forces which impress the same velocity $v$ on each of the $n$ equal particles of which $m$ is composed; so that denoting by $k$ one of these equal forces, we shall have

$$f = nk.$$

"Let $h$ be the force which is capable of impressing the velocity $u$ on each of the equal particles, in the time which is required by the force $k$ to impress upon it the velocity $v$. These forces, acting on the same material particle, will be to each other in the same proportion as $u$ and $v$; and since $v = iu$, we must have

$$k = ih.$$

In like manner we shall have

$$f' = n'k', \qquad k' = i'h',$$

regarding $f'$ as the sum of $n'$ forces ($k'$) capable of impressing the velocity $v'$ on each of the equal particles constituting $m'$, and denoting by $h'$ the force which will impress on each of these particles the velocity $u$. Moreover, $h$ and $h'$ being forces which are capable of impressing in the same instant of time the same velocity $u$ on two particles of equal mass, that is to say, on two of the particles the common mass of which has been represented by $\mu$, it follows from

what precedes that we must have $h=h'$. According to the preceding equations we shall therefore have also,

$$f=inh, \quad f'=i'n'h;$$

and, observing the values of $m, m', v, v'$, we shall be able to conclude the truth of the proportion which it was required to prove.

3. "This being established, let us consider a body of any magnitude and form, all the particles of which move in parallel straight lines, with a common velocity which may vary with the time. Suppose this body divided into an infinite number of material particles equal in mass, such as those which we have just defined. We may attribute the motion of all these particles to forces which will be equal and parallel throughout the whole extent of the body: their resultant, for any portion of the body, will be equal to their sum, and may be supposed to be applied at the centre of gravity of this same portion. The corresponding forces for any two portions will be proportional to their masses; consequently, if we denote by $f$ the total force which acts upon the body, $m$ its mass, and $\phi$ the force which corresponds to a portion of this mass assumed as unity, we shall have

$$f=m\phi.$$

As for the force $\phi$, it will be measured by the velocity generated in the particles of the body in an indefinitely small time; or we may say that it is measured by the velocity generated in a unit of time if the change of velocity be uniform, if otherwise, by that which would be generated in a unit of time supposing the velocity to change at the same rate as at the moment in question.

"The force $f$, which is the resultant or the sum of the indefinitely small forces, which we may suppose to be applied to all the particles of which the body is composed, is called *moving force*; the factor $\phi$ of its value $m\phi$ is called *accelerating force*, and is nothing else than the moving force which acts upon a unit of mass.

"The moving force becomes a *pressure*, when the mass upon which it acts is prevented from moving by a fixed plane perpendicular to its direction. A pressure and a moving force therefore differ from each other only in this, that the indefinitely small velocities which a pressure tends to produce are continually destroyed by the resistance of the fixed plane, whereas those which are actually produced during each instant by the moving force accumulate in the moving body, and after a finite time a finite velocity is produced. Two pressures are to each other as the masses multiplied by the infinitely small velocities which they tend to impress upon them in the same instant of time, and which in fact they would impress upon them if the masses were free.

4. "If the motion common to all the particles of a body be uniformly accelerated, and we call $g$ the increase of velocity which takes place in each unit of time, we have

$$\phi=g, \quad f=mg$$

For another constant force $f'$ acting upon a mass $m'$, and producing a velocity $g'$ in a unit of time, we should have in like manner

$$f'=m'g'.$$

Now observation has shewn that two heavy bodies, whatever be the difference of the substances of which they are composed, acquire the same velocity in falling in vacuum during the same interval of time. In the case of gravity then we have $g=g'$; and, consequently, the weights $f$ and $f'$ of any two bodies are as their masses $m$ and $m'$. The simple fact, testified by daily experience, that heterogeneous bodies have equal weights under different volumes, would not suffice to decide whether their masses are equal or unequal; it is necessary to know besides, that gravity impresses upon them the same motion, to be able to conclude, from the equality of the weights, the equality of the quantities of matter.

"The weight of a heavy body which falls in vacuum is its moving force, and gravity is its accelerating force. For shortness' sake, we usually call the velocity $g$ *gravity*; properly speaking it is only the measure of the force of gravity.

5. "If given forces act at the surface or upon other parts of a solid body, and all its particles have equal and parallel velocities, it follows that these forces must have a single resultant, which will coincide in magnitude and direction with the moving force, as we have just defined it, and from which we can deduce the accelerating force by dividing it by the whole mass of the body.

"Suppose, for example, that a heavy body falls in air, in water, or in any other fluid, and that its form and its density, if it is not homogeneous, are symmetrical round a vertical axis. It is evident that everything being alike round about this axis, all points of the body will describe vertical straight lines; which implies, since the body is supposed solid, that they all have the same velocity at the same instant. The resistance of the medium, which acts on the surface of the body, will be equivalent then to a single force in direction of the axis of figure. Let $R$ denote its intensity at any instant, $\psi$ the corresponding part of the accelerating force of the body, and $m$ its mass; then we shall have

$$\psi = \frac{R}{m}.$$

6. "The same constant force, acting successively upon different masses, will produce uniformly accelerated motions, in which the accelerating force, or the constant increment of the velocity in each unit of time will be inversely proportional to the mass.

"Thus, for example, $f$ being the weight $mg$ of a mass $m$, if we suspend this mass to one extremity of a thread which is attached by its other extremity to another mass $m'$ placed upon a horizontal plane, it is evident that these two masses will receive the same uniformly accelerated motion, which will be that due to the moving force $f$, if we neglect the effect of friction

and of the vertical portion of the thread. If then we call $g'$ the accelerating force of this motion, we shall have

$$g' = \frac{f}{m+m'},$$

or, which is the same thing,

$$g' = g \cos a,$$

$a$ being such an angle that

$$m = (m+m') \cos a.$$

Consequently, the motion in question will be the same as that of a heavy body upon an inclined plane which makes an angle $a$ with the vertical.

"All bodies being moveable, and capable of receiving velocities which are inversely proportional to their masses, when they are submitted during the same length of time to the action of the same force, it follows that there is no such thing as a body actually *fixed;* those which we called fixed are bodies of which the masses are very large in comparison with those upon which the moving forces depend, and which receive consequently from the action of those forces only indefinitely small velocities. At the surface of the earth, these are the bodies which are attached to the surface and which with the terrestrial globe may be considered as making only one mass; and, in fact, taking this mass for the $m'$ of the preceding example, we see that the velocity $g'$, which will be impressed upon it in a unit of time by a weight $mg$ corresponding to a mass $m$ of ordinary magnitude, may be regarded as altogether insensible.

7. "It is usual to employ the term *quantity of motion* to denote the product of a body's mass by its velocity. It would however be more correct to speak of the *quantity of velocity,* since it is the velocity which animates the body, and the motion is only a subsequent effect.

"There is no force which produces instantaneously a finite quantity of motion. The impact of a solid body in motion against a solid body at rest impresses upon it, in a very short space of time, but not infinitely short, a velocity which under some circumstances may be very great; and, during this interval of time, the two bodies are not sensibly displaced. However hard we may suppose them to be, they will be always slightly compressed; the velocity passes from one to the other by infinitely small degrees; and if we neglect all consideration of the elasticity of the bodies, their mutual action will cease as soon as their velocities are equal. This rapid communication of velocity, without any sensible displacement of the masses, is what we call a *percussion* or an *impulse;* it is equivalent, as we see, to a moving force acting during a very short time with a very great intensity.

"Considering an impulse in this manner as the sum of the infinitely small actions of a moving force, we may conclude that it can be resolved into two other impulses, in given directions, according to the rule of the parallelogram of forces, as may each of the successive actions. If, for example, we exert upon the *head* of a wedge a normal impulse which we will call $P$, it will be resolved into two others perpendicular to the *faces* of the wedge; and if we

denote these two components by $Q$ and $Q'$, by $K$ and $K'$ the length of the corresponding faces, and by $H$ that of the head of the wedge, it is easy to see that we shall have, according to the rule given by the parallelogram of forces,

$$Q : P :: K : H,$$
$$Q' : P :: K' : H;$$

whence we have

$$Q = \frac{PK}{H}, \qquad Q' = \frac{PK'}{H}.$$

Thus, supposing that this impulse $P$ arises from a mass in striking the head of the wedge with a velocity $a$, its two faces, or rather the fixed obstacles against which they press, will be under the same conditions as if they were struck normally by the same mass $m$, animated with velocities proportional to their lengths, and expressed by $\frac{Ka}{H}$ and $\frac{K'a}{H}$.

8. "If a solid body at rest be struck at the same instant, in opposite directions, by two other bodies of which the masses are $m$ and $m'$, and the velocities $v$ and $v'$; and if these three bodies be symmetrical around the same axis both as to form and as to density, and if all the particles of the last two move parallel to this straight line, their impulses upon the intermediate body will produce equilibrium, when the quantities of motion $mv$, $m'v'$ are equal, that is to say, these quantities of motion will pass, in the course of a very short time, into the intermediate body, and will destroy each other without sensibly displacing that body.

"The equilibrium will also be produced if we suppress the intermediate body, and allow the communication of velocity to take place directly between the two others. Thus two solid bodies, moving in the same straight line, will be reduced to rest if they impinge upon each other, provided their masses are inversely as their velocities; and, conversely, the products of the masses and velocities are equal, when equilibrium is produced by the impact of two solid bodies. We suppose here, as has been already said, that the two bodies are symmetrical about one and the same straight line, and the velocities of all points of the bodies parallel to this straight line, which is that which passes through the centres of gravity of the two masses. The condition of equilibrium in the impact of such bodies is the equality of their quantities of motion, or the equation

$$mv = m'v',$$

$m$ and $m'$ being their masses, and $v$ and $v'$ their velocities.

"It follows from this law of equilibrium that impact furnishes the most direct means of measuring the mass of a body. Suppose a known velocity $a$ to be impressed upon all points of a body, the mass of which is assumed as unity; then if we could ascertain the velocity $v$, with which all points of another body must be animated, in order that the two bodies impinging

upon each other may be reduced to rest, the mass of this second body would be represented numerically by the ratio $\frac{a}{v}$; but it is unnecessary to remark that this is not a practicable method, and that it is always the *weight* of bodies to which we refer, when we wish to estimate their masses.

"It follows also that two blows, inflicted on a solid body, must be regarded as equivalent, when they correspond to equal quantities of motion; so that in the last example, the head and the two faces of the wedge will experience the same effects, or will be struck with the same energy, if the mass $m$ and the velocity $a$ be replaced by a mass $m'$ and a velocity $a'$, such that $ma = m'a'$.

9. "The science of Dynamics is a continual application of the principles which have been here explained, and of which it is necessary to obtain a distinct apprehension before proceeding to the resolution of problems relative to the motion of bodies."

28. We are now prepared with Dynamical principles, which we shall proceed to apply to the cases, of a particle under the action of uniform forces, and of the collision of bodies. A more general application of the principles would require the use of a higher calculus, than any with which the student is supposed to be acquainted.

## ON THE RECTILINEAR MOTION OF A SINGLE BODY CONSIDERED AS A PARTICLE, UNDER THE ACTION OF A UNIFORM FORCE.

The following is the fundamental proposition of this branch of Dynamics.

29. PROP. *If* s *be the space described from rest, in the time* t, *by a body under the action of a uniform accelerating force,* f, *then* $s = \frac{ft^2}{2}$.

Suppose the time to be divided into $n$ equal intervals, each equal to $\tau$, so that $t = n\tau$; then the velocity of the body at the end of the 1st, 2nd, 3rd...intervals will be $f\tau$,

$2f\tau$, $3f\tau$...respectively; (Art. 15). Now suppose the body, instead of moving as it actually does, to move through the whole of each interval of time with the velocity which it had at the *beginning* of the interval, and let $s_1$ be the space which would be described in the time $t$ on this hypothesis; again, suppose the body to move through each interval with the velocity which it has at the *end* of the interval, and let $s_2$ be the space described on this hypothesis; then

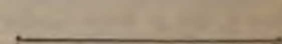
$$s_1 = 0 \,.\, \tau + f\tau \,.\, \tau + 2f\tau \,.\, \tau + \ldots + (n-1) f\tau \,.\, \tau$$
$$= f\tau^2 \frac{n(n-1)}{2} = \frac{ft^2}{2}\left(1 - \frac{1}{n}\right),$$
$$s_2 = f\tau \,.\, \tau + 2f\tau \,.\, \tau + 3f\tau \,.\, \tau + \ldots + nf\tau \,.\, \tau$$
$$= f\tau^2 \frac{n(n+1)}{2} = \frac{ft^2}{2}\left(1 + \frac{1}{n}\right).$$

Now it is manifest that $s$ is intermediate in value to $s_1$ and $s_2$; but when we suppose the interval $\tau$ to be indefinitely small, and $n$ indefinitely great, we bring the two hypothetical cases, which we have supposed, to coincide with the real motion of the body, and in this case $\frac{1}{n}$ is indefinitely small, and

$$s_1 = s_2 = \frac{ft^2}{2};$$

$$\therefore \text{ also, } s = \frac{ft^2}{2}.$$

30. We shall give another proof of the same proposition, which, though in some respects inferior to that which precedes, is worthy of the student's attention on account of its brevity.




Let $A$ be the point from which the body starts, $B$ its place at the end of the time $t$, so that $AB = s$: then the velocity of the body at $B = ft$. Now suppose the body to start from $B$ with a velocity $ft$, and to proceed towards $A$, the force $f$ acting in the same direction as before, and therefore *retarding* its motion: at the end of the time $t$ the body would, if no force

acted, have described a space $ft^2$; also we know that the space, through which the force $f$ would carry it, is $s$; hence the space through which the body will move in the time $t$, when it starts with a velocity $ft$, and is retarded by a force $f$, must be $ft^2 - s$. But at the end of the time $t$ the body will be at $A$, since the force $f$ will destroy the velocity $ft$, in exactly the same space which it required to generate it. Hence we have

$$ft^2 - s = BA = s;$$

$$\therefore\ s = \frac{ft^2}{2}.$$

COR. If $v$ be the velocity of the body at the end of the time $t$, $v = ft$; $\therefore\ s = \dfrac{v^2}{2f}$, or $v^2 = 2fs$. Hence, when a body falls under the action of gravity, this being a uniform force, the velocity $\propto$ the time, and the square of the velocity $\propto$ the space described.

31. If the body, instead of starting from rest, begins to move with a given velocity, the formulæ of the preceding article are easily adapted.

For let $V$ be the initial velocity, $v$ the velocity at the time $t$, $s$ the space described. Then the velocity at the time $t$ is the original velocity $\pm$ that which is generated or destroyed by the force, or

$$v = V \pm ft,$$

the upper or lower sign being used, according as the force accelerates or retards.

And the space will be that, which would have been described by the body moving uniformly with a velocity $V$ $\pm$ that which it would describe under the action of the force only, or

$$s = Vt \pm \frac{ft^2}{2},$$

the upper or lower sign being taken according to the same rule as before.

COR. We can obtain $v$ in terms of $s$: for

$$\begin{aligned} v^2 &= (V \pm ft)^2 \\ &= V^2 \pm 2f\left(Vt \pm \frac{ft^2}{2}\right) \\ &= V^2 \pm 2fs. \end{aligned}$$

32. For distinctness' sake we will here collect in a tabular form the formulæ which have been proved, and which may be applied to a variety of examples.

When the body starts from rest,

$$\left.\begin{aligned} s &= \frac{ft^2}{2} \\ v &= ft \\ v^2 &= 2fs \end{aligned}\right\}\text{............}(A).$$

When the body starts with an initial velocity $V$,

$$\left.\begin{aligned} s &= Vt \pm \frac{ft^2}{2} \\ v &= V \pm ft \\ v^2 &= V^2 \pm 2fs \end{aligned}\right\}\text{.........}(B),$$

the upper or lower sign to be taken, according as $f$ accelerates or retards the motion.

33. These formulæ possess peculiar interest, because they apply to the case of a body falling under the action of gravity. We have before said, that gravity is a uniform force, at least it is sensibly so for small distances above the earth's surface; we may here remark in addition, that in consequence of the earth not being actually spherical, the force of gravity is not exactly the same at all points of the earth's surface, but varies slightly with the latitude; we may consider, however, that for all ordinary purposes its magnitude is measured by the quantity $g = 32.2$ feet. The value of $g$ is not determined by direct experiment, but may be most accurately found by observations of the pendulum, according to principles hereafter to be developed.

34. We shall now illustrate the formulæ ($A$) and ($B$) by some examples of falling bodies.

Ex. 1. Two bodies are let fall at an interval of 1″, to find how far they will be apart after the lapse of 4″ from the fall of the first.

Let $s\, s'$ be the spaces through which they have respectively fallen, then

$$s = \frac{g}{2} 4^2 = \frac{g}{2} \times 16,$$

$$s' = \frac{g}{2} 3^2 = \frac{g}{2} \times 9;$$

$\therefore s - s' = \frac{g}{2} \times 7 = 16.1 \times 7 = 112.7$ feet, the distance required.

Ex. 2. When two bodies are let fall from the same point, but not at the same moment, the distances between them at the end of successive equal intervals of time increase in arithmetical progression.

Let $s\, s'$ be distances, described by the two bodies respectively in the time $t$ from the fall of the first, $\tau$ the interval of time between the starting of the two, then

$$s = \frac{gt^2}{2},$$

$$s' = \frac{g}{2} \{t - \tau\}^2,$$

$$\therefore s - s' = \frac{g}{2} \{2t\tau - \tau^2\};$$

$s - s'$ is the distance between the bodies at the time $t$; call this $d$, and let $d_1$, $d_2$...be the distances corresponding to the times $t + t_1$, $t + 2t_1$;

$$\therefore d = \frac{g}{2} \{2t\tau - \tau^2\},$$

$$d_1 = \frac{g}{2}\{2(t+t_1)\tau - \tau^2\},$$

$$d_2 = \frac{g}{2}\{2(t+2t_1)\tau - \tau^2\},$$

$$\&c. = \&c.;$$

$$\therefore\ d_1 - d = gt_1\tau = d_2 - d_1 = \&c.;$$

that is, the quantities $dd_1d_2\ldots$&c., are in arithmetical progression.

Ex. 3. A stone is thrown into a well, and it is observed that 2″ elapse before the sound of its striking the bottom is heard; neglecting the time occupied by the transmission of the sound, find the depth of the well.

Let $s$ be the depth;

$$\text{then } s = \frac{g}{2}2^2 = 2g = 64.4 \text{ feet.}$$

Ex. 4. A ball is projected upwards with a given velocity; it falls, and on striking the earth rebounds with a loss of $\left(\frac{1}{n}\right)^{\text{th}}$ part of its velocity; find to what height it will rise after any given number of rebounds, and the whole space which will be described by the ball before coming to rest.

Let $V$ be the velocity, then the height to which it rises $= \dfrac{V^2}{2g}$.

When it reaches the earth its velocity is $V$, therefore the velocity of rebound is $V\left(1 - \frac{1}{n}\right)$ by hypothesis, and the height to which it rises $= \dfrac{V^2}{2g}\left(1 - \dfrac{1}{n}\right)^2$.

In like manner, the height to which it rises after the second rebound $= \dfrac{V^2}{2g}\left(1 - \dfrac{1}{n}\right)^4$.

And similarly, it will be seen, that the height after the $p^{\text{th}}$

rebound $= \frac{V^2}{2g}\left(1 - \frac{1}{n}\right)^{2p}$, which is the answer to the first part of the question.

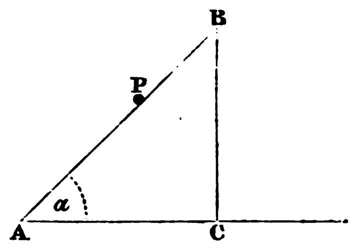
The whole space described by the ball

$$= \frac{V^2}{g}\left\{1 + \left(1 - \frac{1}{n}\right)^2 + \left(1 - \frac{1}{n}\right)^4 + \ldots ad\ infinitum\right\}$$

$$= \frac{V^2}{g}\frac{1}{1 - \left(1 - \frac{1}{n}\right)^2} = \frac{V^2}{g}\frac{n^2}{n^2 - (n-1)^2} = \frac{V^2}{g}\frac{n^2}{2n - 1}.$$

35. The formulæ ($A$) and ($B$) are applicable, with a slight modification, to the case of a body falling down a smooth inclined plane. In this case we must conceive the force of gravity to be resolved into two parts, one parallel to the plane, the other perpendicular to it; the latter will produce pressure on the plane, the former will accelerate the motion of the body, and is the only part with which we shall be here concerned. Let $\alpha$ be the inclination of the plane, then the resolved part of $g$ parallel to the plane will be $g \sin \alpha$, and this quantity must be used for $f$ in our fundamental formulæ.

Ex. 1. To find the velocity acquired by a body in falling down a given inclined plane.




Let $AB$ be the inclined plane, $\alpha$ its inclination, $P$ the place of the body at a given time, $BP = s$, $v =$ the velocity at $P$; then we have by our formula,

$$v^2 = 2g \sin \alpha \times s,$$

which gives the velocity. If we draw $BC$ perpendicular to the horizontal line through $A$, and call $V$ the velocity at $A$, we have

$$V^2 = 2g \,.\, AB \sin \alpha$$

$$= 2g \,.\, BC.$$

Hence the velocity is the same at $A$, as if the body had fallen through the vertical space $BC$: that is to say, the velocity generated by gravity, depends upon the vertical space through which it is allowed to act; a result, which might perhaps have been anticipated, and which was in fact assumed by Galileo.

Ex. 2. A body is projected upwards, along an inclined plane, with a given velocity, find how high it will ascend, and the time of ascent.

If $v$ be the velocity at the time $t$, when it has ascended through a space $s$, $\alpha$ the angle of the plane, and $V$ the given velocity of projection, we have

$$v^2 = V^2 - 2gs \sin \alpha;$$

when $v = 0$ the body will stop, and the distance required will be given by the equation,

$$s = \frac{V^2}{2g \sin \alpha}.$$

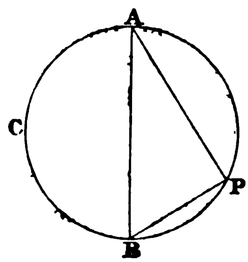
For the time, we have

$$v = V - gt \sin \alpha,$$

and the body stops when

$$t = \frac{V}{g \sin \alpha}.$$

Ex. 3. Let $ACB$ be a circle in a vertical plane, $A$ its highest point; the time of descent down all chords drawn through $A$, considered as inclined planes, will be the same.




Let $AP$ be any chord, $\alpha$ its inclination to the horizon; draw the vertical diameter $AB$, and join $BP$, then $ABP = 90^\circ - BAP = \alpha$: now if $t$ be the time of descent down $AP$, we shall have

$$AP = g \sin \alpha \frac{t^2}{2};$$

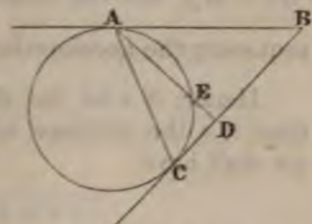
$$\text{but, } AP = AB \sin \alpha,$$

$$\therefore AB = \frac{gt^2}{2},$$

$$\text{and } t = \sqrt{\frac{2AB}{g}}, \text{ which is independent}$$

of $\alpha$, and is therefore the same for all chords.

Ex. 4. From a given point to draw the line, down which as an inclined plane a particle will descend to a fixed line in the shortest time possible.




Let $A$ be the given point, and through it draw a horizontal line to meet the given line in $B$; in the given line take $BC$ equal to $BA$; join $AC$, which shall be the line required.

This will easily appear from the fact that a vertical circle having $A$ for its highest point will *touch* the given straight line in $C$; for if $AC$ be not the line of quickest descent, let $AED$ cutting the circle just mentioned in $E$ be the required line; then the time of descent down $AC$ is equal to the time down $AE$, and therefore is less than the time down $AD$. Therefore $AC$ is the line of quickest descent.

And it may be shewn, that in general the line of quickest descent from a point to any given curve will be found, by describing a circle, to touch the horizontal line passing through the given point at that point, and also to touch the given curve. The chord joining the points of contact will be the line of quickest descent.

## ON THE MOTION OF TWO FALLING BODIES CONNECTED BY A STRING.

36. We have distinguished problems of this class from those which we have been recently engaged in considering, because in those we were able to regard gravity only as an

accelerating force, whereas in these we must consider the mass moved, and deduce the accelerating force by the third Law of Motion.

Ex. 1. Suppose we have two bodies, the masses of which are $M$ and $M'$ respectively, connected by a fine string passing over a smooth peg or small pully $A$, the only effect of which is to change the direction of the string.


Let $M$ be greater than $M'$; then the moving force producing motion is the difference of the weights, or $Mg - M'g$, and the whole mass moved is $M + M'$, consequently the accelerating force is $\dfrac{M - M'}{M + M'}g$.

Hence, if $x$ be the distance of $M$ from $A$ at the time $t$, $a$ the distance when the motion commenced, we shall have

$$x = a + \frac{M - M'}{M + M'}\frac{gt^2}{2}.$$

Let it be required to determine the tension of the string.

To do this we observe, that, if the weight $Mg$ were suspended at the extremity of a string and were at rest, the tension would be $Mg$, the accelerating force $g$ being entirely employed in producing tension. As it is, however, a portion of the accelerating force acting on $M$ is employed in producing motion, and the remainder in producing tension of the string; now the former portion we have determined to be $\dfrac{M - M'}{M + M'}g$, hence

$$\text{the tension} = M\left\{g - \frac{M - M'}{M + M'}g\right\}$$

$$= \frac{2MM'}{M + M'}g;$$

an expression, which, it may be observed, involves $M$ and $M'$ symmetrically, as manifestly it ought.

This problem may be solved in another manner, which will give us at once the tension of the string and the accelerating force.

Thus, let $T$ be the tension of the string: then the moving force upon the body $M$ will be $Mg - T$, and the accelerating force therefore $g - \frac{T}{M}$. In like manner the accelerating force upon $M'$ will be $g - \frac{T}{M'}$; and these two must be equal, but of opposite algebraical signs, since one of the bodies necessarily *ascends* with the same velocity with which the other *descends*.

Therefore, $$g - \frac{T}{M} = -g + \frac{T}{M'},$$

$$T\left(\frac{1}{M} + \frac{1}{M'}\right) = 2g,$$

$$\text{and } T = \frac{2MM'}{M+M'}g, \text{ as before;}$$

and the accelerating force

$$= g - \frac{T}{M} = g - \frac{2M'}{M+M'}g = \frac{M-M'}{M+M'}g.$$

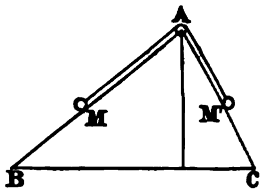
Ex. 2. Let us take a numerical illustration of the last example.

Suppose $M = 2M'$, and $a = 0$, to find how far $M$ will descend in 1″. We have,

$$s = \frac{2-1}{2+1}\frac{g}{2} = \frac{16.1}{3} = 5.3 \text{ feet.}$$

Also in this case the tension of the string $= \frac{2}{3}Mg =$ two-thirds of the heavier weight.

Ex. 3. Two weights are placed upon two opposite inclined planes, and connected by a fine string which passes over a small pully at the highest point of the planes; to determine the motion.




Let $AB$, $AC$ be the two planes, $\alpha$, $\beta$ their respective inclinations.

Then the part of the weight $Mg$ which is effective in producing motion is $Mg \sin \alpha$, and that of $M'g$ is $M'g \sin \beta$; the difference of them or $Mg \sin \alpha - M'g \sin \beta$ is the moving force; the mass moved is $M + M'$; therefore the accelerating force is

$$\frac{M \sin \alpha - M' \sin \beta}{M + M'} g.$$

If $x$ be the distance of $M$ from $A$ at the time $t$, $a$ the distance at the beginning of the motion,

$$x = a + \frac{M \sin \alpha - M' \sin \beta}{M + M'} \frac{gt^2}{2}.$$

The tension of the string may be found as in Ex. 1.

Ex. 4. A numerical illustration of the preceding.

Suppose $M = 3M'$, $\alpha = 30^0$, $\beta = 60^0$: then,

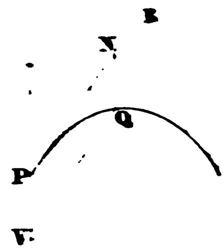
$$x = a + \frac{\frac{3}{2} - \frac{\sqrt{3}}{2}}{3 + 1} \cdot \frac{gt^2}{2}$$

$$= a + \frac{3 - 1.7}{16} gt^2 = a + 2.6t^2, \text{ nearly.}$$

## ON PROJECTILES.

37. We now come to the case of motion not rectilinear; and the principal problem, to which the formulæ already established are applicable, is that of the motion of a projectile, that is, of a heavy body which has been projected in a direction not vertical. The results which we obtain, it may be observed, are not practically correct, since we omit the consideration of the resistance of the air, and suppose the body to move in a perfect vacuum.

38. *The path of a projectile will be a parabola.*




It is evident that the path will be in one plane: let it lie in the plane of the paper, and let $P$ be the point of projection, $PNR$ the line in which the body is projected, which will manifestly be a tangent to the curve described.

Let $V$ be the velocity of projection, and $N$ the point at which the body would arrive with this velocity in the time $t$, so that $PN = Vt$.

From $N$ draw $NQ$ vertical, and make $NQ = \frac{gt^2}{2}$; then, since the space, which the body would describe in the time $t$ under the action of gravity only, is $NQ$, and if gravity had not acted the body would have been at $N$, therefore when the body is simultaneously animated by its original velocity $V$, and that generated by gravity, it will be at $Q$.

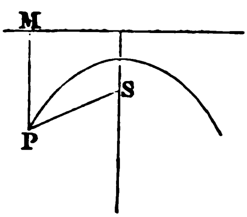
Complete the parallelogram $PVQN$, then

$$PV = NQ = \frac{gt^2}{2},$$

$$QV = PN = Vt;$$

$$\therefore QV^2 = V^2t^2 = \frac{2V^2}{g}.PV.$$

But in the parabola $QV^2 = 4SP.PV$. (See Conic Sections, Prop. IX. page 169.) Hence $Q$ lies in a parabola, of which the axis is vertical, and the distance of $P$ from the directrix or focus is $\frac{V^2}{2g}$.

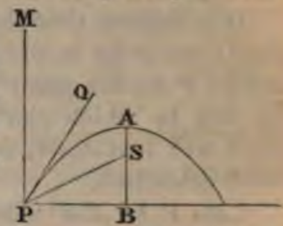



COR. From the point $P$ draw $PM$ perpendicular to the directrix; then

$$V^2 = 2g.SP = 2g.PM.$$

Hence, *the velocity at any point of the parabola is that which would be acquired in falling from the directrix.*

39. *To determine the position of the focus of the parabola.*




Let $P$ be the point of projection, $PQ$ the direction of projection, $ASB$ the axis of the parabola, $PM$ vertical, $PB$ horizontal; and let $QPB$ (which is called the angle of projection) $= a$.

From $P$ draw $PS$, making the same angle with $PQ$ as $PQ$ makes with $PM$; then, by a property of the parabola, the point $S$, in which $PS$ meets the axis, is the focus.

$$\text{Now } SPB = QPB - QPS = QPB - MPQ$$

$$= a - 90^0 + a = 2a - 90^0;$$

$$\therefore PB = SP \sin 2a, \quad SB = -SP \cos 2a,$$

$$\text{but } SP = \frac{V^2}{2g}; \therefore PB = \frac{V^2}{2g} \sin 2a, \quad SB = -\frac{V^2}{2g} \cos 2a^*.$$

The lines $PB$, $SB$, determine the position of $S$; for we have only to measure the distance $PB$ horizontally, and $BS$ vertically, and we shall determine $S$. Or we may determine the position of the line $PS$ as above, and take in it a point the distance of which from $P$ is $\frac{V^2}{2g}$; this will be the focus.

Cor. If $a = 45^0$, the focus of the parabola is in the horizontal plane passing through the point of projection.

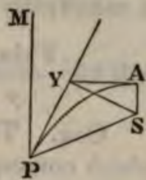
40. *To determine the greatest height to which the projectile will rise.*

The velocity of projection may be supposed to be resolved into two velocities, one horizontal $= V \cos a$, the other vertical $= V \sin a$. The vertical motion of the body will be the same as if it were projected vertically with this latter velocity; hence, if $h$ be the height required, we have

$$h = \frac{V^2 \sin^2 a}{2g}.$$

* The quantity $-\frac{V^2}{2g} \cos 2a$ will be *positive*, because in the figure $2a$ has been supposed to be greater than $90^0$.

41. *To find the latus rectum of the parabola.*




Draw the perpendicular $SY$ on the tangent $PY$, which is the line of projection; let $A$ be the vertex, then $AY$ is a tangent at the vertex, and therefore horizontal. (See Conics, Prop. VI. page 167.)

Then $SYA = 90^0 - \alpha$, and $SPY = MPY = 90^0 - \alpha$;

$$\therefore \text{the latus rectum} = 4AS = 4SY \cos\alpha = 4SP\cos^2\alpha$$

$$= \frac{2V^2}{g} \cos^2\alpha.$$

42. *To find the range of the projectile, that is, the distance from the point of projection at which the body meets the horizontal plane through the point of projection, and the time of flight.*

The range will evidently be twice the distance of the point of projection from the axis of the parabola. But this distance was shewn in Art. 39 to be $\frac{V^2}{2g}\sin 2\alpha$;

$$\therefore \text{the range} = \frac{V^2}{g}\sin 2\alpha;$$

To find the time of flight we have only to observe, that the horizontal velocity of the body is uniform and equal to $V\cos\alpha$, and therefore that the time which elapses between the moment of projection and the moment of the body striking the ground is that occupied by a body moving through the space $\frac{V^2}{g}\sin 2\alpha$ with the velocity $V\cos\alpha$;

$$\therefore \text{the time of flight} = \frac{V^2\sin 2\alpha}{gV\cos\alpha} = \frac{2V}{g}\sin\alpha.$$

The same result may be arrived at from the consideration of the vertical motion of the body. For since the vertical velocity of projection is $V\sin\alpha$, the time of flight will be that which elapses before a body projected vertically with a velocity $V\sin\alpha$ falls and strikes the ground; and this time

is manifestly expressed by $\frac{2V\sin\alpha}{g}$, since the time of ascent will be $\frac{V\sin\alpha}{g}$ and the time of falling the same.

COR. The greatest value which $\sin 2\alpha$ can have is 1, which corresponds to $\alpha = 45^\circ$; hence, the range of a projectile will be greatest when the angle of projection is $45^\circ$.

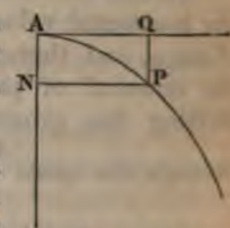
43. The preceding are the principal propositions respecting projectiles. We shall now give a few examples, which may be multiplied to any extent.

Ex. 1. A body is projected horizontally with a velocity of 4 feet per second, to find the latus rectum of the parabola described.

The general expression for the latus rectum is $\frac{2V^2}{g}\cos^2\alpha$; and in this case $\alpha = 0$, and $V = 4$;

$$\therefore \text{ the latus rectum} = \frac{16}{16.1} = 1 \text{ foot, nearly.}$$

When the body is projected horizontally, it is manifest that the point of projection is the vertex of the parabola, and the general investigation of the path becomes much simplified.




Let $AQ$ be the direction of projection, and let $Q$ be the point at which the body would arrive from $A$ in the time $t$ with a velocity $V$; $\therefore AQ = Vt$.

Again, take $QP = \frac{gt^2}{2}$, so that $QP$ is the distance through which the body would fall in time $t$ under the action of gravity only: then $P$ is the actual place of the body at the time $t$.

Complete the rectangle $ANPQ$; then

$$PN = Vt,$$

$$AN = \frac{gt^2}{2};$$

$$\therefore\ PN^2 = V^2t^2 = \frac{2V^2}{g} \cdot AN;$$

but in the parabola $PN^2 = 4AS \cdot AN$ (See Conics, Prop. V. page 167);

$$\therefore\ 4AS = \frac{2V^2}{g}.$$

Ex. 2. A body is projected at an angle of 30°, with the velocity which it would acquire in falling through 5 feet; find the range.

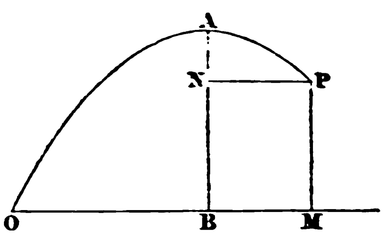
In general, the range $= \dfrac{V^2}{g} \sin 2\alpha$,

and in this case, $\qquad V^2 = 2g \times 5;$

$$\therefore\ \frac{V^2}{g} = 10.$$

and the range $= 10 \sin 60° = \sqrt{3}$ feet.

Ex. 3. To find the relation between the velocity and angle of projection, in order that the projectile may strike a given point.




Let $O$ be the point of projection, $OM$ horizontal, and $MP$ vertical; let $P$ be the given point, then it will be given provided we know $OM$ and $MP$: let $OM = h$, $MP = k$. Draw $AB$ the axis of the parabola, and $PN$ perpendicular to $AB$.

Then we have proved, (Arts. 39, 40) that $OB = \dfrac{V^2 \sin 2\alpha}{2g}$,

$$\text{and } AB = \frac{V^2 \sin^2 \alpha}{2g};$$

$$\therefore\ AN = \frac{V^2 \sin^2 \alpha}{2g} - k,\ PN = h - \frac{V^2 \sin 2\alpha}{2g};$$

but the latus rectum $= \dfrac{2V^2}{g} \cos^2 \alpha$;

$$\therefore PN^2 = \frac{2V^2}{g}\cos^2\alpha \,.\, AN;$$

$$\text{or } \left(h - \frac{V^2 \sin 2\alpha}{2g}\right)^2 = \frac{2V^2}{g}\cos^2\alpha \left(\frac{V^2\sin^2\alpha}{2g} - k\right);$$

$$\text{or } h^2 - 2h\frac{V^2}{g}\sin\alpha\,\cos\alpha + 2k\frac{V^2}{g}\cos^2\alpha = 0,$$

which is the relation between $\alpha$ and $V$ required.

Suppose, for instance, that $\alpha = 45^0$, $h = 90$, $k = 9$, then, to find $V$, we have

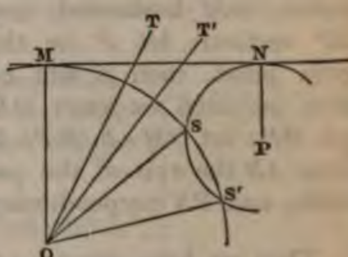
$$h^2 - k\frac{V^2}{g} + k\frac{V^2}{g} = 0,$$

$$\frac{V^2}{g} = \frac{h^2}{h-k} = \frac{90^2}{81} = 100,$$

$$\therefore V = 10\sqrt{g}$$

$= 57$ feet per second, nearly.

Ex. 4. Given the velocity of projection, to construct for the direction in order that the projectile may pass through a given point.




Let $O$ be the point of projection; $P$ the point through which the projectile is to pass; $V$ the velocity of projection.

Draw $OM$ vertical and equal to $\frac{V^2}{2g}$; $MN$ horizontal, which will be the directrix; $PN$ perpendicular to the directrix; with centres $O$ and $P$, distances $OM$ and $PN$, describe two circles; if these do not intersect the problem is impossible; if they do, let $S$ and $S'$ be their points of intersection; $S$ and $S'$ will be the foci of two parabolas, either of which may be the path taken by the projectile. Join $OS$, $OS'$; bisect $MOS$, $MOS'$ by $OT$, $OT'$; these will be the directions of projection required.

The demonstration is obvious.

In the particular case in which the two circles touch each other, there will be only one direction of projection. In this case it will be easily seen that we have,

$$(OM + NP)^2 = (OM - NP)^2 + MN^2,$$

$$\text{or}\quad 4OM \cdot NP = MN^2,$$

that is, according to the notation adopted in the preceding example,

$$\frac{2V^2}{g}\left(\frac{V^2}{2g} - k\right) = k^2 \text{.........}(A)$$

Let us, for the sake of illustration, introduce this condition into the equation of the preceding example: that equation may be written thus,

$$2k\frac{V^2}{g}\sin\alpha\cos\alpha + 2k\frac{V^2}{g}\sin^2\alpha = k^2 + 2k\frac{V^2}{g},$$

$$= \frac{V^4}{g^2}\text{by }(A).$$

$$\therefore \left(\frac{V^2}{g} - 2k\sin^2\alpha\right)^2 = 4k^2\sin^2\alpha\cos^2\alpha$$

$$= 4\sin^2\alpha\,(1 - \sin^2\alpha)\left(\frac{V^4}{g^2} - \frac{2V^2}{g}k\right);$$

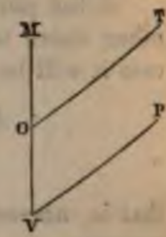
$$\therefore 4\sin^4\alpha\left(\frac{V^4}{g^2} - \frac{2V^2}{g}k + k^2\right) - 4\sin^2\alpha\,\frac{V^2}{g}\left(\frac{V^2}{g} - k\right) + \frac{V^4}{g^2} = 0,$$

$$\text{or}\quad 2\sin^2\alpha\left(\frac{V^2}{g} - k\right) - \frac{V^2}{g} = 0,$$

$$\therefore 2\sin^2\alpha = \frac{V^2}{V^2 - gk},$$

or the equation gives only *one* value for $\alpha$, which is in accordance with the result before announced.

Ex. 5. The converse of the preceding; that is, given the direction to construct for the velocity.




Let $O$ be the point of projection; $P$ the point through which the projectile is to pass; $OT$ the given direction. Draw $MOV$ vertical; $PV$ parallel to $OT$. Take $OM$ such that $PV^2 = 4\,OM \cdot OV$, that is, take $OM$ a third proportional to $OV$ and $\frac{PV}{2}$; then the horizontal line through $M$ is the directrix, and

$$V^2 = 2g \cdot OM.$$

Ex. 6. An attempt is made by using *sights* of different elevations to take account of the parabolic path of a rifle-ball. The formula of Ex. 3, will enable us to calculate the proper elevation of the sight.

It will be understood that the sight of which we speak is a point near the lock of a gun, elevated above the barrel, so that when the line joining this point with the sight at the extremity of the barrel is directed towards the mark, the line of the barrel does in reality point *above* the mark so as to allow of the ball dropping before it strikes. Let $\phi$ be the angle which the elevation of the sight above the barrel subtends at the extremity of the gun; then $\alpha$ is the real angle of elevation of the gun, $\alpha - \phi$ the elevation of the line of sight, and consequently we must have $k = h \tan(\alpha - \phi)$.

Now resume the formula of Ex. 3, which may be written thus,

$$\frac{gh^2}{2V^2} = \cos\alpha\,(h\sin\alpha - k\cos\alpha)$$

$$= h\cos\alpha\,\{\sin\alpha - \cos\alpha\tan(\alpha - \phi)\}$$

$$= h\cos\alpha\frac{\sin\phi}{\cos(\alpha - \phi)} = \frac{h\sin\phi}{\cos\phi + \sin\phi\tan\alpha},$$

but $k(1 + \tan\alpha\tan\phi) = h\tan\alpha - h\tan\phi$,

$$\therefore \tan\alpha = \frac{k + h\tan\phi}{h - k\tan\phi},$$

$$\therefore \cos\phi + \sin\phi\tan\alpha = \frac{\cos\phi\,(h - k\tan\phi) + \sin\phi\,(k + h\tan\phi)}{h - k\tan\phi}$$

$$= \frac{h}{\cos\phi\,(h - k\tan\phi)},$$

$$\therefore \frac{gh^2}{2V^2} = \sin\phi\cos\phi\,(h - k\tan\phi),$$

$$= h\sin\phi\cos\phi - k\sin^2\phi.$$

This is the accurate equation for calculating $\phi$, but since in practice $V$, the velocity of the ball, is enormously great, $\phi$ is very small, and the term $k\sin^2\phi$ is therefore practically insensible; omitting this term, we have the very neat equation,

$$\sin 2\phi = \frac{gh}{V^2},$$

which gives a value of $\phi$ independent of $k$; in other words, the adjustment of the sight depends only upon the horizontal distance of the mark, and is independent of the height of the mark above the ground.

The result of this example may be expressed by saying, that the angle $\phi$ is the angle of elevation proper for the gun, in order to strike a point in the horizontal plane and at the same horizontal distance as the given point.

## MOTION OF A PARTICLE ON A CURVE.

44. When a heavy particle moves on a plane curve, the plane being supposed to be vertical, a portion of the weight of the body will be employed in producing pressure on the curve, and the other portion in producing motion. Hence, in considering the motion of a body on a curve, we suppose the force of gravity at any point to be resolved into two parts, one acting along the *tangent* of the curve, the other along the *normal;* the former is the part which produces motion, and is the only part with which we shall generally be concerned. We have already considered a particular case of motion on a curve, when we treated of a body falling down an inclined plane; but in that case the inclination of the plane being always constant, the force producing motion was uniform, whereas in the more general problem of motion on any

curve the intensity of the force varies from point to point. Consequently the formulæ which we have hitherto used are inapplicable; we shall be able however to give one proposition, which we can demonstrate by general reasoning. We shall recur to the subject hereafter.

45. *When a heavy body falls down the surface of a smooth curve, the velocity at any point is that due to the vertical height through which the body has fallen.*

In the first place we observe, that the pressure of the curve is always perpendicular to the direction of the motion, and therefore destroys no velocity.

In the next place, the proposition is true for an inclined plane, as has been shewn. (See Ex. 1, page 297.)

And, lastly, we may consider the curve to be a succession of inclined planes, the planes being indefinitely short in length and great in number, and the proposition being true for each will be true for all, and therefore for the curve. For though it may be argued, that there is always a loss of velocity in passing from one plane to another, yet we know that, when the planes are so far increased in number and diminished in magnitude as to be considered coincident with the curve, this will not be the case, since by our first observation no velocity is lost*.

Cor. Hence, if a body fall down a smooth curve, as for instance in the interior of a hemispherical bowl, it will after passing the lowest point rise to the same vertical height as that from which it fell. Practically the extent of oscillation will be continually diminished by friction, until the body is brought to rest at the lowest point.

* The proof of this proposition is sometimes given more at length, as in Whewell's *Elementary Treatise on Mechanics*, but I am not aware that the reasoning is thereby made more satisfactory. The real difficulty in passing from the case of a succession of inclined planes to that of a continuous curve is the neglect of the loss of velocity which necessarily takes place in passing from each inclined plane to the next; the student will gain help from observing how it is demonstrated in page 317, that in the case of a circle considered as a polygon of an infinite number of sides no velocity is in this manner destroyed.

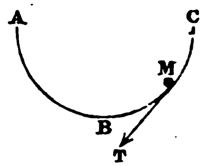
46. There is one point connected with the motion of a body on a curve, of which we can give a general explanation, and on which it is desirable to have distinct notions; and that is, the existence of what is called *centrifugal force*. The explanation is applicable, not only to the motion of a body constrained to move upon a curve line or curve surface, but also to that of a body describing any path not rectilinear under the action of any forces.

When a body, acted upon by any force, moves in the direction in which that force acts, it has a tendency at each moment to proceed only with the velocity which it has at that moment; this follows from the first law of motion, or is the consequence of the *inertia* of the body. But when a body, acted upon by any force, moves transversely to the direction of the force, it has a tendency at each moment, not only to proceed with the velocity with which it is then animated, but also to continue to move in the direction in which it is moving at the instant in question; this also is a consequence of the first law of motion. The motion therefore may be conceived of as though the body were under the action of a force, always tending to make the body leave the path which it actually describes; the force which measures this tendency, is called the *centrifugal force*, a name liable to much objection, because there is in fact no force acting on the body to draw it out of its path, the tendency to leave that path being the result of its own motion only.

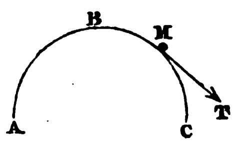
Hence if we conceive the forces, acting on a body which describes a plane curve, to be resolved into two, one in the direction of the tangent, the other in the direction of the normal, we may say that the former portion is expended in accelerating or retarding the body's motion, the latter in changing its direction, or (in other words) in counteracting the *centrifugal force*. And if the body be constrained to move on a curve, the normal portion will be employed in counteracting the centrifugal force, and also in producing pressure on the curve. Therefore, when a heavy particle moves upon a curve in a vertical plane, the pressure on the curve is not the resolved part of the weight in the direction of the normal, as would be the case if the particle were at

rest, but is greater or less than that resolved part according as the centrifugal force tends to increase or diminish the pressure.

Suppose, for instance, a particle $M$ to be falling down the interior of a smooth hemispherical bowl $ABC$ in a vertical plane; draw $MT$ a tangent to the semicircle, then $MT$ is the direction in which the body is moving, and in which it would continue to move if unrestrained, and if gravity were to cease to act; it is evident then that the motion of the body in this case causes a pressure upon the bowl, since it tends to make it move in a direction in which it cannot move in consequence of the interposition of the bowl. In this case therefore we should say, that the pressure was increased by the centrifugal force.




But if the particle $M$ be moving on the exterior of a semicircular curve $ABC$, and if $MT$ be the direction of its motion at any time, then similar considerations shew that the motion of the body tends to diminish the pressure; and in this case there will be a certain point at which the pressure due to the weight of the body will be entirely counteracted by the centrifugal force, and when the body has reached this point it will leave the curve and describe a parabola.




A familiar instance of the action of centrifugal force is the case of a stone thrown with a sling; omitting the consideration of the action of gravity, this reduces itself to the problem of a body attached to one extremity of a string the other extremity of which is fastened; the body will revolve with a uniform velocity in a circle, and the tension of the string, which will depend upon the rate of the body's motion and the length of the string, will measure the intensity of the centrifugal force. We say that the velocity will be uniform, because the tension of the string, (which is the only force acting upon the body) being always in a direction perpendicular to the direction of the body's motion, it can have no effect either to increase or diminish the velocity. If the string be suddenly cut, the body goes off in the direction of the tangent of the

circle in which it has previously moved, and with the velocity which it had previously.

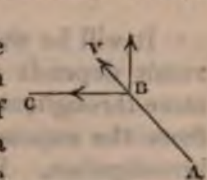
We have said that the notion of centrifugal force applies to the case of free curvilinear motion, as well as the motion of a body constrained by a curve. In this case the centrifugal force will be measured by the resolved part of the force, impressed upon the body, in the direction of the normal to its path. Let us take the example of the projectile. Let $S$ be the focus, $P$ the position of the projectile, $PT$ the tangent, (see figure page 167), then the accelerating force in the direction of the normal

$$= g \sin STP,$$

$$= g\frac{SY}{ST} = g\frac{SY}{SP} = g\frac{AS}{SY};$$

or the centrifugal force, estimated as a moving force, will be $Mg\dfrac{AS}{SY}$, or will vary inversely as the perpendicular from the focus on the tangent.

We will give one case of constrained motion in which the centrifugal force may be determined; the method which we shall adopt is peculiar, and the result will be better understood hereafter; it may however be worthy of the student's attention. The case is that of a body revolving uniformly in a circle, either in consequence of being attached to a string the tension of which will measure the centrifugal force, or in consequence of being constrained to move in a circular groove the pressure upon which will take the place of the tension of the string. The latter supposition will be the more convenient, because we can arrive at it by first supposing the body to move in a groove of the form of a regular polygon, which we can afterwards make a circle, in the same way as in Trigonometry we found the circumference and area of a circle by first finding those of a regular polygon.




Let $AB$, $BC$ be two adjacent sides of the polygon, and let $\theta$ be the angle between them which will also be the angle which a side of the polygon subtends at the centre. Let a side of the polygon $= a$, the radius of the circumscribed circle $= r$, and the number of sides $n$.

Now suppose a body of mass $M$ to move along the side $AB$ of the groove with a velocity $V$; resolve the velocity $V$ into $V\cos\theta$ along the side $BC$ and $V\sin\theta$ perpendicular to it; then it is this latter velocity, which being destroyed by the side $BC$ causes the pressure upon the groove, and the sum of all such pressures throughout the motion will be the measure of the whole pressure sustained by the groove. Assuming then the velocity $V$ to remain unaltered, we shall have for the sum of all the momenta destroyed in the course of one revolution of the body $nMV\sin\theta$.

$$\text{Now}\qquad nMV\sin\theta = 2nMV\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= nMV\,\text{chd}\,\theta\,.\cos\frac{\theta}{2}$$

$$= nMV\times\frac{a}{r}\cos\frac{\theta}{2}.$$

But when the number of sides of the groove is indefinitely increased $na = 2\pi r$, and $\cos\frac{\theta}{2} = 1$; and the above expression becomes $2\pi MV$. Now let $P$ be the pressure which we desire to find; then the time in which the body revolves being $\frac{2\pi r}{V}$, the momentum destroyed by the groove in that time will be $\frac{2\pi r}{V}\,.\,P$; hence we have,

$$\frac{2\pi r}{V}P = 2\pi MV,$$

$$\text{or}\quad P = M\frac{V^2}{r}.$$

It will be observed that the correctness of the preceding result depends entirely upon the fact of $V$ remaining the same throughout the motion; let us prove that this is the case, from the expression given for the velocity in the preceding investigation. We found that along the side $BC$ the velocity would be $V\cos\theta$, similarly along the next side it would be

$V\cos^2\theta$, and so on, and on completing the revolution the velocity would be $V\cos^n\theta$.

$$\text{Now, } \cos\theta = 1 - 2\sin^2\frac{\theta}{2} = 1 - \tfrac{1}{2}\,\text{chd}^2\,\theta,$$

$$= 1 - \frac{a^2}{2r^2},$$

$$\therefore \cos^n\theta = \left(1 - \frac{a^2}{2r^2}\right)^n$$

$$= 1 - \frac{na^2}{2r^2} + \frac{n(n-1)}{1.2}\left(\frac{a^2}{2r^2}\right)^2 + \&c.$$

But when the number of the sides of the polygon is indefinitely increased $na$ becomes $2\pi r$, and therefore

$$n^2a^2 = 4\pi^2r^2,$$

$$\text{or } \quad \frac{na^2}{2r^2} = \frac{2\pi^2}{n};$$

$$\therefore \cos^n\theta = 1 - \frac{2\pi^2}{n} + \frac{\frac{1}{n}\left(\frac{1}{n} - 1\right)}{1.2}(2\pi^2)^2 - \&c.$$

$$= 1, \text{ when } n \text{ is indefinitely great;}$$

hence at the conclusion of a revolution, the velocity $= V$ as at first. We were therefore justified in assuming, that the velocity would remain the same throughout the motion.

Let us take a numerical example of the formula above demonstrated.

Ex. A ball weighing 1lb., attached to a string the length of which is 3 feet, makes a revolution in 4 seconds; to find the tension of the string.

$$\text{We have } \quad \frac{P}{Mg} = \frac{V^2}{gr};$$

$\therefore$ if we estimate the tension in lbs., the number of lbs. required will be $\frac{V^2}{gr}$. Now $\frac{2\pi r}{V}$ = the time of revolution,

$$\therefore \frac{2\pi r}{V} = 4,$$

$$\text{and } r = 3, \therefore V = \frac{3\pi}{2},$$

$$\text{and } \frac{V^2}{gr} = \frac{1}{32.2} \times \frac{1}{3} \times \frac{9\pi^2}{4} = \frac{3\pi^2}{128.8} = .229882 \text{ lbs.}$$

It is not difficult to perceive, that the formula, which we have demonstrated for the centrifugal force in a circle when the velocity is uniform, is also true when the velocity is variable; the quantity $V$ will in this case be no longer a constant quantity. For instance, if we have a particle falling in the interior of a hemispherical bowl; when it reaches the bottom $V^2 = 2gr$ by Art. 45: hence the pressure due to the centrifugal force is $2Mg$, and the whole pressure is $3Mg$ or three times the weight of the particle.

Again, if we take the case (before referred to) of a particle falling down upon the convex surface of a vertical circle, we can determine the actual point at which the body will leave the curve; for let the particle descend from the highest point $B$, let $O$ be the centre of the circle, $M$ the position of the body when its velocity is $V$, and let $BOM = \theta$; then

$$V^2 = 2g(r - r\cos\theta);$$

also the resolved part of gravity in the direction of the normal is $g\cos\theta$; the particle will leave the curve when the centrifugal force is just equal to the pressure due to gravity, that is, when

$$2g(1 - \cos\theta) = g\cos\theta,$$

$$\text{or } \cos\theta = \tfrac{2}{3},$$

which determines the point required.

The principles which have been developed explain certain results which at first sight appear paradoxical: thus a pail of water may be made to revolve in a vertical circle, so that the water shall not leave the pail; the only condition to be satis-

fied is, that if $r$ be the radius of the circle in which the pail revolves, $V$ the velocity at the highest point,

$$\frac{V^2}{r} > g.$$

This curious result is exhibited conveniently by the centrifugal railway, which consists of a rail bent into a spiral form and lying as nearly as possible in a vertical plane, one extremity being much higher than the other; a car descends from the higher extremity and ascending in virtue of the velocity acquired follows the spiral rail, the passenger in the car having his head downwards whenever the rail assumes the form of an arc having the concavity downwards. It is not difficult to calculate the height from which it is necessary to descend, in order to make the passage of such a portion of the railway safe; suppose the arc to be circular and its radius $r$, also let $x$ be the height of the starting point of the car above the highest point of the concave portion of the rail, then, omitting all consideration of friction, the square of the velocity at that point will be $2gx$; therefore the centrifugal force will be $\frac{2gx}{r}$; and this must be greater than the force of gravity,

$$\therefore \text{ we must have } \frac{2gx}{r} > g,$$

$$\text{or } x > \frac{r}{2}.$$

This condition determines the smallest elevation which can be safely given to the upper extremity of the railway.

## ON THE COLLISION OR IMPACT OF BODIES.

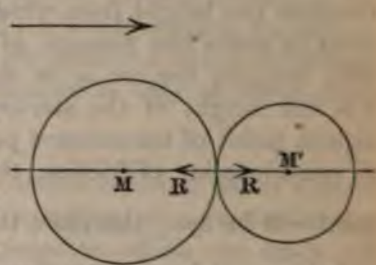
47. In investigating the circumstances of motion which attend the collision of bodies, there are two cases for us to consider; (1) that of *inelastic* bodies, (2) that of *elastic*. We must define the meaning of these terms.

We have already, when speaking of impulsive force, described the nature of the action which takes place when two

elastic bodies impinge upon each other; now if when two bodies impinge upon each other, after the compression of their figures due to the impact has ceased, a force of restitution of figure comes into operation, the bodies are called *elastic*, but if there is no such force, then they are called *inelastic*. To take examples, ivory and glass are highly elastic substances, a lump of putty or clay is perhaps as free from elasticity as any substance.

Let us consider the impact of inelastic bodies.

48. *Two inelastic balls, moving in the same direction but with different velocities, impinge upon each other; to determine the motion after impact.*




Let $MM'$ be the masses of the two balls, $VV'$ their velocities before impact.

When the bodies impinge, there will be an impulsive pressure between them, which we will call $R$, and the value of which it must be our business to find. This pressure will of course be equal in magnitude and opposite in its direction upon the two balls, i.e. accelerating one and retarding the other.

The momentum of the ball $M$ before impact is $MV$, therefore its momentum after impact is $MV - R$, and therefore its velocity $V - \frac{R}{M}$.

Similarly, the velocity of the ball $M'$ after impact is

$$V' + \frac{R}{M'}.$$

But since, by the hypothesis of inelasticity, there is no force after impact to separate the balls, they will proceed with a common velocity;

$$\therefore V - \frac{R}{M} = V' + \frac{R}{M'};$$

$$\therefore R = \frac{MM'}{M+M'}(V-V'),$$

and the common velocity

$$= V - \frac{M'}{M+M'}(V-V') = \frac{MV+M'V'}{M+M'}.$$

If the balls are moving in *opposite* directions, we have only to write $-V'$ instead of $V'$.

COR. If we call $v$ the velocity after impact, we have

$$Mv + M'v = MV + M'V';$$

that is to say, the sum of the momenta of the balls is the same after impact as before it.

This result might have been concluded at once from general reasoning. For when one body impinges upon another, it can only lose momentum by generating momentum equal to that which it loses; this follows from the third Law of Motion; and hence it is evident that no momentum can be lost by the impact. And this reasoning applies not only to inelastic bodies, but to elastic, because whatever be the nature of the dynamical action no momentum can be lost by one body without generating an equal amount in the other.

The same thing will hold of any number of balls, moving in any directions: that is, if $M$, $M'$ ... be the masses of any number of balls, $V$, $V'$ ... their velocities before impact, $v$, $v'$, ... after impact, we must have,

$$Mv + M'v' + \ldots = MV + M'V' + \ldots$$

49. We can now proceed to the problem of elastic bodies. We may consider the impact as consisting of two parts, viz. during the compression of the bodies, and during the restitution of their forms: as long as compression continues, the problem is precisely the same as if the bodies were inelastic, and if we call the impulsive pressure between them during compression $R$, the value of $R$ will be that already found on the supposition of the bodies being inelastic; for, though the bodies do not, for any sensible

time after impact, move with the same velocity, yet during that very short time in which the compression takes place they do so; hence the *force of compression* is already determined. When the restitution of form takes place, a new force is brought into action, which we shall distinguish as the *force of restitution*, and shall call $R'$; to determine $R'$ we must have recourse to experiment, and it is found that the ratio of $R'$ to $R$ is independent of the velocity of the bodies, and dependent only on the nature of the substances of which they are composed. So that, if we make $R' = eR$, we may consider $e$ to be a known quantity, since in any given example, if the substance of the bodies is given, the value of $e$ may be found from experiment, or by reference to tables of elasticity. The quantity $e$ is called the *modulus of elasticity*; it is always some quantity less than 1. The limiting case of $e$ being actually $= 1$ is that of *perfect elasticity*, but there is no such case in nature.

50. *Two elastic balls moving in the same direction, but with different velocities, impinge upon one another; to find the velocities after impact.*

Let $MM'$ be the masses of the bodies,

$VV'$ their velocities before impact,

$vv'$ .................. after .........

$RR'$ the forces of compression and restitution respectively, so that the whole impulsive force between the balls $= R + R' = R(1 + e)$, where $e$ is the modulus of elasticity.

We may find $R$ on the supposition of the bodies being inelastic; hence by our previous investigation (Art. 48),

$$R = \frac{MM'}{M + M'}(V - V');$$

$$\therefore v = V - \frac{R + R'}{M} = V - (1 + e)\frac{R}{M} = V - (1 + e)\frac{M'}{M + M'}(V - V'),$$

$$v' = V' + \frac{R + R'}{M'} = V' + (1 + e)\frac{R}{M'} = V' + (1 + e)\frac{M}{M + M'}(V - V').$$

If the balls are moving in opposite directions we must change the sign of $V'$, as in Art. 48.

Cor. We have, by subtraction,

$$v - v' = V - V' - (1 + e)(V - V'),$$
$$= -e(V - V').$$

If we suppose $V$ to be greater than $V'$, and that after the impact the ball $M'$ is driven on by $M$ in the direction in which it was moving before impact, $v'$ will be greater than $v$, and we may write the preceding equation thus,

$$\frac{v' - v}{V - V'} = e.$$

Now $V - V'$ is the *relative* velocity of the balls before impact, that is, the rate at which they approach each other, and $v' - v$ is the relative velocity after impact, or the rate at which they separate; hence the preceding formula may be expressed by saying, that the ratio of the relative velocities before and after impact is a quantity depending only on the nature of the substances of which the balls are composed. This is a law which may be easily made the subject of direct observation, and from it conversely may be deduced the law which has been enunciated in Art. 49, and which has been made the basis of our investigations.

For we have

$$v = V - \frac{R + R'}{M},$$

$$v' = V' + \frac{R + R'}{M'};$$

but by experiment,

$$v - v' = -e(V - V'),$$

$$\therefore -e(V - V') = (V - V') - (R + R')\frac{M + M'}{MM'},$$

$$\text{or } R + R' = (1 + e)\frac{MM'}{M + M'}(V - V'),$$

$$\text{but } R = \frac{MM'}{M + M'}(V - V'),$$

$$\therefore R' = e\frac{MM'}{M + M'}(V - V') = eR.$$

51. *An elastic ball impinges directly upon a fixed plane; to find the velocity after impact.*

Let $V$ be ball's velocity before impact,

$v$ ..................... after .........

$RR'$ the forces of compression and restitution,

$e$ the modulus of elasticity.

Then, to find $R$, we suppose the body inelastic; but in this case there would be no velocity after impact, since the plane is fixed;

$$\therefore V - \frac{R}{M} = 0, \text{ or } R = MV;$$

$$\therefore R + R' = (1 + e) MV,$$

$$\text{and } v = V - (1 + e) V = -eV.$$

Hence the ball's velocity will be diminished in the ratio of $1 : e$. The negative sign indicates that the motion after impact must be in the opposite direction to that before impact, which must manifestly be the case.

52. By *oblique* impact we intend to express those cases of impact, in which the direction of the velocity does not coincide with the direction of the mutual impulsive pressure.

53. *A body impinges upon a fixed plane, in the direction of a line making a given angle with the normal to the plane; to determine the motion after impact.*

Let $V$ be the velocity before impact, $\alpha$ the angle which its direction makes with the normal to the plane: $v$, $\theta$ similar quantities after impact. The rest of the notation as before.

We may suppose the velocity $V$ to be resolved into two velocities, one parallel to the plane ($V \sin \alpha$), the other perpendicular to it ($V \cos \alpha$); the former will not be altered by the impact, the latter may be treated as in the case of direct impact, and will therefore be diminished in the ratio of $1 : e$. The resolved parts of the velocity after impact, parallel and

perpendicular to the plane, are $v \sin \theta$, and $v \cos \theta$ respectively; hence we shall have,

$$v \sin \theta = V \sin \alpha,$$

$$v \cos \theta = -eV \cos \alpha;$$

$$\therefore \cot \theta = -e \cot \alpha, \text{ and } v^2 = V^2 (\sin^2 \alpha + e^2 \cos^2 \alpha),$$

which equations determine $\theta$ and $v$.

It may be observed that this investigation is applicable to the case of impact on any surface, by substituting for the plane on which the impact has been supposed to take place the plane which touches the surface at the point of impact.

Cor. If the elasticity be perfect, or $e = 1$, we shall have

$$\cot \theta = -\cot \alpha,$$

$$\text{or } \theta = -\alpha,$$

$$\text{and } v^2 = V^2, \text{ or } v = V.$$

The interpretation of these results is, that the ball will rebound from the plane with a velocity equal to that of incidence, and in a direction making an angle with the normal equal to the angle of incidence, but on the opposite side of the normal.

54. The more general case of the oblique impact of two balls may be solved in like manner by resolving the velocity of each ball into two, namely, one in the direction of the mutual impulsive pressure, and the other in the direction at right angles to it; then the latter portions of the velocities will not be affected by the impact, and the former will be modified exactly in the same way as if the impact had been direct.

We shall subjoin a few examples of impact.

Ex. 1. *A perfectly elastic ball impinges directly upon another, and this upon a third; compare the velocity communicated to the third, with that which would have been communicated if the first had impinged upon it.*

Let $M M' M''$ be the masses of the balls, $V$ the velocity of $M$ before impact.

Let $R$ be the force of compression, then we should find, by an investigation such as that in Art. 48, that

$$R = \frac{MM'}{M + M'} V;$$

$\therefore$ the velocity of $M'$ after impact

$$= \frac{2R}{M'} = \frac{2M}{M + M'} V, \text{ (since } e = 1).$$

In like manner, the velocity communicated to $M''$

$$= \frac{2M'}{M' + M''} \cdot \frac{2M}{M + M'} V.$$

But the velocity, which would have been communicated, if $M$ had impinged upon $M''$,

$$= \frac{2M}{M + M''} V;$$

$$\therefore \text{ the ratio required is } \frac{2M'(M + M'')}{(M' + M'')(M + M')}.$$

Ex. 2. *In the direct impact of perfectly elastic bodies, the sum of the masses of the bodies multiplied each by the square of its velocity is the same before and after impact.*

Let $MM'$ be the masses of the bodies, $VV'$ their respective velocities before, and $vv'$ after, impact.

Then, we have seen (Art. 50), that

$$v = V - \frac{2M'}{M + M'} (V - V'),$$

$$v' = V' + \frac{2M}{M + M'} (V - V'),$$

since $e = 1$:

$$\therefore v - v' = V - V' - 2(V - V') = -(V - V'),$$
$$\text{or } v + V = v' + V' \ldots\ldots\ldots\ldots\ldots (1).$$

Again,

$$Mv + M'v' = MV + M'V',$$
$$\text{or } M(v - V) = -M'(v' - V') \ldots\ldots\ldots (2).$$

Multiplying together (1) and (2), we have

$$M(v^2 - V^2) = -M'(v'^2 - V'^2),$$
$$\text{or, } Mv^2 + M'v'^2 = MV^2 + M'V'^2.$$

The mass of a body multiplied by the square of its velocity is called its *Vis Viva;* hence it appears, that when the elasticity is perfect, the sum of the *Vis Viva* of two impinging bodies is not altered by impact.

Ex. 3. *In the collision of imperfectly elastic bodies, Vis Viva is lost by the impact.*

In this case we have

$$v = V - (1 + e)\frac{M'}{M + M'}(V - V'),$$

$$v' = V' + (1 + e)\frac{M}{M + M'}(V - V');$$

$$\therefore Mv + M'v' = MV + M'V';$$

also $v - v' = V - V' - (1 + e)(V - V') = -e(V - V');$

$$\therefore (Mv + M'v')^2 = (MV + M'V')^2,$$

and $MM'(v - v')^2 = MM'e^2(V - V')^2$

$$= MM'(V - V')^2 - (1 - e^2)MM'(V - V')^2;$$

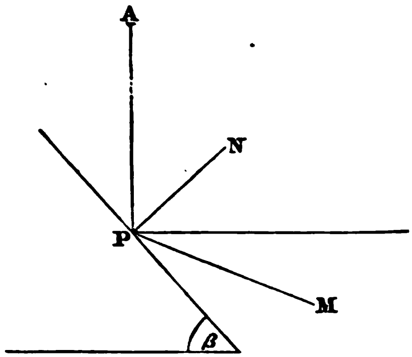
$\therefore$ by addition, $(M + M')(Mv^2 + M'v'^2) = (M + M')(MV^2 + M'V'^2)$

$$- (1 - e^2)MM'(V - V')^2,$$

or $Mv^2 + M'v'^2 = MV^2 + M'V'^2 - (1 - e^2)\dfrac{MM'}{M + M'}(V - V')^2;$

which proves the proposition, since $e$ is less than 1.

Ex. 4. *An elastic ball falls from a given height up given inclined plane; to find the latus rectum of the para described after the rebound.*




Let $\beta$ be the angle of the plane: $A$ the point from which the body falls; $P$ the point of impact, $AP = h$. Let $PN$ be normal to the plane, $PB$ horizontal, and $PM$ the direction in which the ball goes off after impact; also let $BPM = \theta$.

Then the velocity with which the body reaches $P = \surd$ and the angle $APN = \beta$, hence if $V$ be the velocity with w the ball leaves the plane,

$$V^2 = 2gh\,(\sin^2\beta + e^2\cos^2\beta). \quad \text{(Art. 53.)}$$

Again, $NPM = 90^\circ - \beta + \theta$, hence we have, (by the article,)

$$\tan(\beta - \theta) = e\cot\beta.$$

The latus rectum of the parabola described

$$= \frac{2V^2}{g}\cos^2\theta. \quad \text{(Art. 41.)}$$

And we have

$$\frac{\tan\beta - \tan\theta}{1 + \tan\beta\tan\theta} = \frac{e}{\tan\beta},$$

$$\tan^2\beta - \tan\theta\tan\beta = e + e\tan\beta\tan\theta;$$

$$\therefore \tan\theta = \frac{\tan^2\beta - e}{\tan\beta\,(1 + e)},$$

$$\text{and } \cos^2\theta = \frac{1}{1 + \tan^2\theta} = \frac{\tan^2\beta\,(1 + e)^2}{\tan^2\beta\,(1 + e)^2 + (\tan^2\beta - e)^2}$$

$$= \frac{\tan^2\beta\,(1 + e)^2}{\sec^2\beta\,(\tan^2\beta + e^2)} = \sin^2\beta\,\frac{(1 + e)^2}{\tan^2\beta + e^2};$$

$$\cos^2\theta = 4h(\sin^2 B - e^2\cos^2 B)\sin^2 B\,\frac{1-e^2}{\sin^2 B - e^2}$$

$$= 4h\sin^2 B\cos^2 B\,(1-e^2)$$

$$= h\sin^2 2B\,(1-e^2).$$

5. *To determine the motion of the centre of gravity of two elastic balls after direct impact.*

At the time $t$, let $x$, $x'$, $x_1$ be the distances of the two balls and their centre of gravity respectively from any fixed point in the direction of motion. The rest of the notation as before, Art. 43, page 3[illegible].

$$x_1 = \frac{Mx + M'x'}{M + M'} = \frac{Mv + M'v'}{M + M'}\,t$$

$$= \frac{MV + M'V'}{M + M'}\,t.$$

Hence the centre of gravity will move with the uniform velocity $\frac{MV + M'V'}{M + M'}$, or with the velocity which it had before impact. If the motion be such that the centre of gravity is at rest before impact, then it will be at rest after.

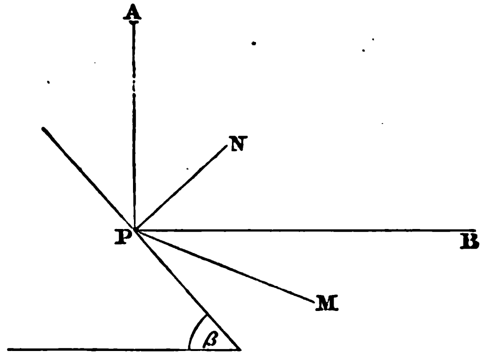
The same method of demonstration may be applied to the case of any number of balls.

6. *A perfectly elastic ball $M$ impinges directly upon $M'$ and this upon a third $M''$; to find the relation between the masses of the balls in order that the velocity communicated to $M''$ may be the greatest possible.*

By Ex. 1, we have already found the expression for the velocity communicated to $M''$, namely $\frac{4MM'}{(M' + M'')(M + M')}V$, where $V$ is the velocity of $M$. It will be necessary therefore to make $\frac{MM'}{(M' + M'')(M + M')}$ as great as possible, or

Ex. 4. *An elastic ball falls from a given height upon a given inclined plane; to find the latus rectum of the parabola described after the rebound.*

Let $\beta$ be the angle of the plane: $A$ the point from which the body falls; $P$ the point of impact, $AP = h$. Let $PN$ be normal to the plane, $PB$ horizontal, and $PM$ the direction in which the ball goes off after impact; also let $BPM = \theta$.




Then the velocity with which the body reaches $P = \sqrt{2gh}$, and the angle $APN = \beta$, hence if $V$ be the velocity with which the ball leaves the plane,

$$V^2 = 2gh(\sin^2\beta + e^2\cos^2\beta). \quad \text{(Art. 53.)}$$

Again, $NPM = 90^0 - \beta + \theta$, hence we have, (by the same article,)

$$\tan(\beta - \theta) = e\cot\beta.$$

The latus rectum of the parabola described

$$= \frac{2V^2}{g}\cos^2\theta. \quad \text{(Art. 41.)}$$

And we have

$$\frac{\tan\beta - \tan\theta}{1 + \tan\beta\tan\theta} = \frac{e}{\tan\beta},$$

$$\tan^2\beta - \tan\theta\tan\beta = e + e\tan\beta\tan\theta;$$

$$\therefore \tan\theta = \frac{\tan^2\beta - e}{\tan\beta(1+e)},$$

$$\text{and } \cos^2\theta = \frac{1}{1+\tan^2\theta} = \frac{\tan^2\beta(1+e)^2}{\tan^2\beta(1+e)^2 + (\tan^2\beta - e)^2}$$

$$= \frac{\tan^2\beta(1+e)^2}{\sec^2\beta(\tan^2\beta + e^2)} = \sin^2\beta\,\frac{(1+e)^2}{\tan^2\beta + e^2};$$

$$\therefore \frac{2V^2}{g}\cos^2\theta = 4h(\sin^2\beta + e^2\cos^2\beta)\sin^2\beta\,\frac{(1+e)^2}{\tan^2\beta + e^2}$$
$$= 4h\sin^2\beta\cos^2\beta\,(1+e)^2$$
$$= h\sin^2 2\beta\,(1+e)^2.$$

Ex. 5. *To determine the motion of the centre of gravity of two elastic balls after direct impact.*

At the time $t$, let $x$, $x'$, $x_1$, be the distances of the two balls and of their centre of gravity respectively from any fixed point in the direction of motion. The rest of the notation as before. Then, by Art. 43, page 246,

$$x_1 = \frac{Mx + M'x'}{M + M'} = \frac{Mv + M'v'}{M + M'}\,t,$$
$$= \frac{MV + M'V'}{M + M'}\,t.$$

Hence the centre of gravity will move with the uniform velocity $\frac{MV + M'V'}{M + M'}$, or with the velocity which it had before impact. If the motion be such that the centre of gravity is at rest before impact, then it will be at rest after impact.

The same method of demonstration may be applied to the collision of any number of balls.

Ex. 6. *A perfectly elastic ball* M *impinges directly upon another* M′ *and this upon a third* M″, *to find the relation between the masses of the balls in order that the velocity communicated to the last may be the greatest possible.*

In Ex. 1, we have already found the expression for the velocity communicated to $M''$, namely $\frac{2MM'}{(M' + M'')(M + M')}\,V$, where $V$ is the velocity of $M$. It will be necessary therefore to make $\frac{MM'}{(M' + M'')(M + M')}$ as great as possible, or

$\dfrac{(M' + M'')(M + M')}{MM'}$ as small as possible. This we shall do by the method given in the note on page 45.

Let $$x = \frac{(M' + M'')(M + M')}{MM'},$$

$$\therefore M'^2 + (M + M'')M' + MM'' = xMM',$$

$$M'^2 - (Mx - M - M'')M' + \frac{(Mx - M - M'')^2}{4}$$
$$= \frac{(Mx - M - M'')^2}{4} - MM'';$$

$$\therefore M' = \frac{Mx - M - M''}{2} + \sqrt{\frac{(Mx - M - M'')^2}{4} - MM''}.$$

The smallest value that $x$ can have is that which makes

$$(Mx - M - M'')^2 = 4MM'',$$
$$\text{or } M' = \sqrt{MM''};$$

that is, the mass of the middle ball must be a mean proportional between the masses of the first and third.

# NEWTON.

## SECTION I.

[DIGRESSION CONCERNING THE CURVATURE OF CURVE LINES.]

## SECTION II.

## SECTION III.

[APPENDIX CONTAINING THE THEORY OF CYCLOIDAL OSCILLATIONS.]

# NEWTON.

## INTRODUCTORY REMARKS.

1. It may perhaps assist the student towards the right understanding of those extracts from Newton's Principia which follow, if we preface them with a few remarks respecting their general purpose.

2. The ordinary processes of geometry and trigonometry are sufficient for the mensuration, and for the discussion of the properties, of straight lines and figures bounded by straight lines; but these methods fail when we come to the consideration of curve lines and figures bounded by curves. We require then some method, which shall enable us to extend our calculations to this more difficult case, and such a method is propounded and developed by Newton: he considers that although a figure inclosed by a curve line is not a polygon, yet a polygon may be made to approach as near as we please to such a figure by increasing the number and diminishing the length of its sides; according to a common phraseology, a figure inclosed by a curve line may be regarded as the *limit* of a polygon, and thus we are enabled to extend to the former propositions proved concerning the latter.

We have already, in fact, anticipated this view in more than one case, as for instance, in determining the area and circumference of a circle, (Trig. Art. 51, page 148); for we deduced those expressions by observing the values to which the area and circumference of a regular polygon continually approximated, when the number of the sides was increased and their length diminished indefinitely*.

We also anticipated the principle, when we regarded the tangent of a curve as the line drawn through two points in the curve, when one of those points is made to move up indefinitely near to the other; in other words, we regarded

* Examine also Arts. 56—58, pages 154—156.

the tangent as the limiting position of the secant, when the points through which the secant is drawn coalesce. (Conics, page 161).

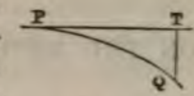
3. Having used the term *limit*, let us endeavour to define it strictly.

Def. The limit of the value of a quantity is the value to which the quantity continually approaches, (though it never reaches it,) and from which it may be made to differ by less than any assignable quantity, when any element on which the quantity depends is indefinitely increased or diminished.

The limiting position of a line would be similarly defined. Hence it appears, that we do not assert that a quantity ever is equal to its limit, but only that it continually approximates to it. Thus we do not say that a circle is a polygon, or a tangent a secant, but only that a polygon continually approximates to a circle as the number of its sides is increased and their length diminished, and that a secant continually approaches to a tangent as the points of section approach each other.

4. The difficulty, which we have pointed out as existing in the application of mathematics to Geometry, exists also in the application of them to Mechanics. Thus we have seen, that in our treatise on Dynamics we were restricted to the consideration of uniform force, because we had no calculus which enabled us to treat of force varying from one moment to another. We may however consider, that if we suppose the case of a number of impulses, and suppose these impulses to be indefinite in number but also indefinitely small in intensity, we shall have a hypothetical system of impulses approximating as near as we please to the character of continuous varying force; in other words, a continuous varying force may be regarded as the *limit* of a system of impulses.

5. The method of calculation, which Newton has founded on this idea of a *limit*, and which he has developed in the first section of his Principia, he calls *The method of prime and ultimate ratios.* The propriety of the name will be seen by considering an example.




Let us suppose $PQ$ to be a portion of a curve, $PT$ the tangent to it at $P$, $TQ$ perpendicular to $PT$: then we may consider the curve $PQ$ to be traced out by a point, moving according to some given law, and $PT$ to be traced out by a point, which moves in the direction in which the former point was moving at $P$. Now it will be hereafter proved, that the limit of the ratio of the lines $PT$ and $PQ$, when $TQ$ is drawn indefinitely near to $P$, is one of equality; hence, if we regard $PQ$ and $PT$ as described by two points beginning to move from $P$, we may speak of their *nascent* state, and say that their *prime* ratio (that is, the ratio which they have *at first*,) is one of equality; or, on the other hand, we may suppose $PQ$ and $PT$ to be continually diminished by the approach of $TQ$ to $P$, and then we may speak of their *vanishing* state and say, that their *ultimate* ratio is one of equality. *Prime* and *ultimate* therefore are, in general, expressions for the same thing, contemplated from two different points of view.

6. It may be well to observe, that when Newton speaks of two quantities being *ultimately equal*, he does not mean that they ever are really equal, but that they are tending to the same limit; thus, to take an algebraical illustration, according to Newton's phrase, $a + x$ and $a + 2x$ are ultimately equal when $x$ is indefinitely diminished, because both tend to the same limit, viz. the quantity $a$.

And, in like manner, in the example adduced in the preceding article, Newton would speak of $PT$ and $PQ$ being ultimately equal; not asserting thereby that those lines are ever really equal, but only that they constantly tend to equality, and that the difference between their ratio and unity diminishes indefinitely as the line $TQ$ approaches $P$.

In the Scholium at the conclusion of the first section Newton himself considers some of the objections which may be raised against his method.

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NOTE. In the following version of the three sections, some demonstrations and propositions have been introduced which are not found in the Principia: all such interpolations are marked by being inclosed in parentheses.

## SECTION I.

### ON THE METHOD OF PRIME AND ULTIMATE RATIOS.

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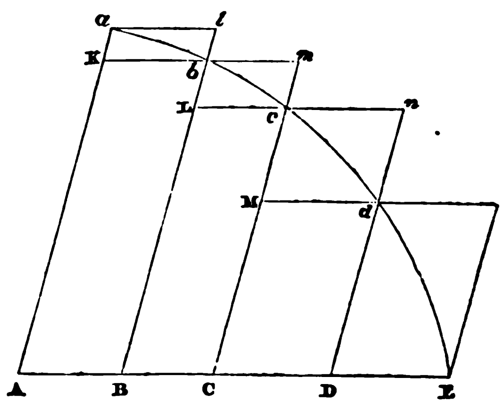
### LEMMA I.

*Quantities and the ratios of quantities which tend constantly to equality, and may be made to approximate to each other by less than any assignable difference, become ultimately equal.*

For if not, let them become ultimately unequal, and their difference be ultimately $D$. Therefore they cannot approximate to each other by less than the difference $D$, and this is contrary to the hypothesis, which is, that they *may* approximate by less than any assignable difference. Wherefore they do not become ultimately unequal, that is, they become ultimately equal. Q.E.D.

### LEMMA II.

*If in any figure* AaE, *bounded by the straight lines* Aa, AE *and the curve* acE, *there be inscribed any number of parallelograms* Ab, Bc, Cd......*on equal bases* AB, BC, CD...... *and sides* Bb, Cc......*parallel to the side of the figure* Aa, *and the parallelograms* aKbl, bLcm, cMdn......*be completed: then, if the breadth of these parallelograms be diminished and*



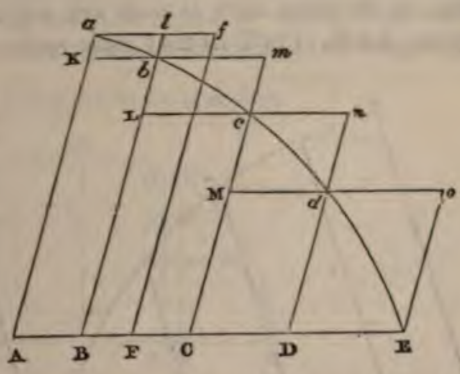
*their number increased indefinitely, the ultimate ratios of the inscribed figure* AKbLcM............, *the circumscribed figure* AalbmcndoE, *and the curvilinear figure* AabcdE, *will be ratios of equality.*

For the difference of the inscribed and circumscribed figures is the sum of the parallelograms *Kl*, *Lm*, *Mn*, *Do*......, that is, (since the bases are all equal) the parallelogram *AalB*. But this parallelogram, by diminishing its breadth indefinitely, may be made less than any assignable quantity. Therefore, by Lemma I., the inscribed and circumscribed figures, and *à fortiori* the curvilinear figure which is intermediate to the two, become ultimately equal. Q.E.D.

## LEMMA III.

*The same ultimate ratios are also ratios of equality, when the breadths of the parallelograms* AB, BC, CD,...... *are unequal, and all are diminished indefinitely.*

For let *AF* be equal to the greatest breadth, and complete the parallelogram *AafF*. Then this parallelogram will be greater than the difference between the inscribed and circumscribed figures; but, when its breadth is diminished indefinitely, it will become less than any assignable quantity, and therefore *à fortiori* the difference between the inscribed and circumscribed figures will be less than any assignable quantity. Hence, as in the preceding Lemma, the ultimate ratios of the inscribed, the circumscribed, and the curvilinear figure, will be ratios of equality. Q.E.D.



COR. 1. Hence the ultimate sum of the evanescent parallelograms coincides with the curvilinear figure.

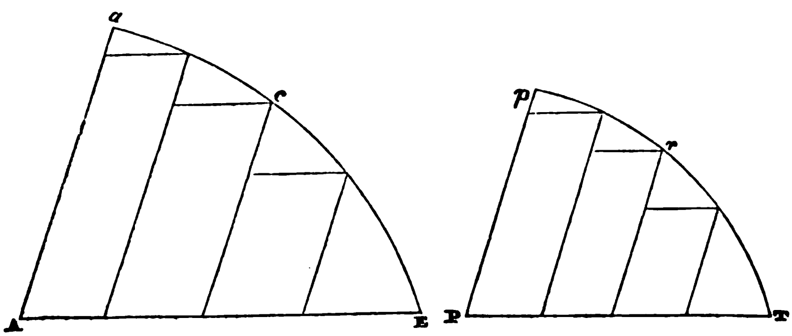
COR. 2. And *à fortiori* the rectilinear figure, which is included by the chords of the evanescent arcs *ab*, *bc*, *cd*, &c. coincides ultimately with the curvilinear figure.

COR. 3. As in like manner does the circumscribed figure which is included by the tangents of the same arcs.

COR. 4. And the perimeters of these ultimate figures are not rectilinear, but the curvilinear limits of rectilinear perimeters.

LEMMA IV.

*If in two figures* AacE, PprT, *are inscribed two series of parallelograms, (as in the preceding Lemmas,) the number in the two series being the same, and if when the breadths of the parallelograms are diminished and their number increased indefinitely, the ultimate ratios of the parallelograms in one figure to those in the other each to each are all the same; then are the figures* AacE, PprT *in that same ratio.*




For as the parallelograms are each to each, so (*componendo*) is the sum of all in one figure to the sum of all in the other, and therefore the figure *AacE* to the figure *PprT*; for by the preceding Lemma, the ultimate ratio of these

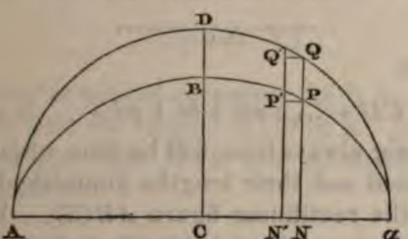
figures to the series of inscribed parallelograms is a ratio of equality*. Q.E.D.

COR. Hence if two quantities of any kind be divided into the same number of parts; and those parts, when the number of them is increased and their magnitude diminished indefinitely, have a given ratio, namely, the first to the first, the second to the second, and so on, the whole quantities will be to each other in that same given ratio. For if in the figures of this Lemma the parallelograms be taken having the same ratio to each other as the parts, the sums of the parts will always be as the sums of the parallelograms; and therefore, when the number of the parts and parallelograms are increased and their magnitude diminished indefinitely, they will be in the ultimate ratio of parallelogram to parallelogram, that is, (by hypothesis) in the ultimate ratio of part to part.

[DEF. One curvilinear figure is said to be similar to another, when any rectilinear figure being inscribed in the first, a similar rectilinear figure may be inscribed in the other.

In other words, similar curvilinear figures are the *limits* of similar rectilinear figures, the sides of which have been indefinitely increased in number and diminished in length.]

* [By means of this Lemma we may find the area of an ellipse.




For if $ABa$ be the ellipse, $ADa$ the auxiliary circle, and we describe in these a series of parallelograms on equal bases, such as $PP'N'N$, $QQ'N'N$, these parallelograms are to each other as $PN : QN$, or as $BC : AC$, (Conics, Prop. VI. p. 178.)

Therefore

$$\text{area of ellipse} : \text{area of circle} :: BC : AC,$$

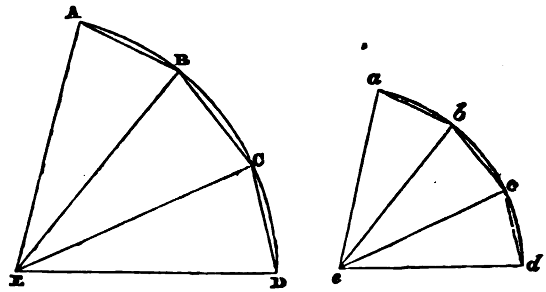
$$\text{or, area of ellipse} = \frac{BC}{AC} \cdot \pi AC^2$$

$$= \pi AC \cdot BC.]$$

## LEMMA V.

*The homologous sides of similar curvilinear figures are proportional, and their areas are in the duplicate ratio of the sides**.

[Let $AED$, $aed$ be two similar curvilinear figures, of which the sides $AE$, $ED$, $AD$, are homologous to $ae$, $ed$, $ad$, respectively; then, by definition, if $ABCDE$ be a polygon inscribed in one, a similar polygon $abcde$ may be inscribed in the other.




Join $EB$, $EC$..., $eb$, $ec$..., dividing the polygons into the same number of similar triangles;

$$\therefore AB : ab :: AE : ae,$$

$$\text{similarly, } BC : bc :: BE : be :: AE : ae,$$

$$CD : cd :: AE : ae,$$

....................

$\therefore$ *componendo*,

$$AB + BC + CD + \ldots : ab + bc + cd + \ldots :: AE : ae.$$

Now this, being always true, will be true when the number of sides is increased and their lengths diminished indefinitely; but, in this case, the rectilinear figure $ABCD$... becomes ultimately equal to the curve line $AD$, and $abcd$... to $ad$;

$$\therefore AD : ad :: AE : ae :: ED : ed.$$

* [Newton gives this Lemma without any demonstration; that given in the text is in fact merely an expansion of the assertion that similar curvilinear polygons are the *limits* of similar rectilinear polygons, and the mind which has grasped the principle that what is true of two polygons is necessarily true of the limits of those polygons, will at once receive the Lemma, as Newton has given it, without formal demonstration.]

Again, polygon $EABCD$ : polygon $eabcd$ :: $AE^2 : ae^2$, and this being always true will be true in the limit as before; therefore, (Lemma III. Cor. 2.)

$$\text{curvil}^r.\ \text{figure } AED : \text{curvil}^r.\ \text{figure } aed :: AE^2 : ae^2$$
$$:: AD^2 : ad^2$$
$$:: ED^2 : ed^2$$

Q.E.D.

Cor. If $AED$, $aed$, are similar figures, and $EC$, $ec$ equally inclined to $ED$, $ed$, then $EC : ec :: ED : ed$.]

[A *curve* may be conceived as being traced by a point, the direction of the motion of which is continually changing.

The *tangent* at any point of a curve, thus considered, is the straight line, in which the generating point would move, if instead of changing the direction of its motion it moved on in the direction which it had at the given point.

A curve is said to be one of *continued curvature*, when the change of direction is not abrupt, but gradual; that is, if $ACB$ (fig. Lemma VI.) be an arc of continued curvature, $AD$ a tangent at $A$, and $BT$ a tangent at $B$, then as the point $B$ moves to $A$ the angle $BTD$ which determines the direction of its motion diminishes, not abruptly, but gradually, and ultimately vanishes.]

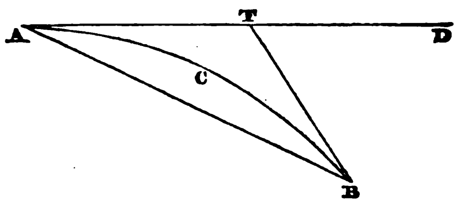
## Lemma VI.

*If any arc of continued curvature* ACB *be subtended by the chord* AB, *and have the tangent* ATD *at* A; *then if the point* B *move up to* A, *the angle* BAD *will diminish indefinitely and ultimately vanish**.

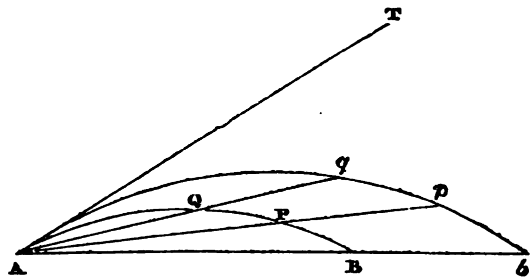
[Draw the tangent $BT$ at $B$; then, since the curvature is continued, the angle $BTD$ continually diminishes as $B$ approaches $A$, and ultimately vanishes; therefore *à fortiori* the

* [It will be easily seen, that the mode of viewing the tangent to a curve, which has been here adopted, coincides with that according to which the tangent is considered as the limiting position of the secant, (see page 161); for, since the angle $BAD$ ultimately vanishes, the secant $AB$ ultimately coincides with the tangent $AD$.]

angle $BAT$, which is less than $BTD$, continually diminishes and ultimately vanishes*. Q.E.D.




COR. Similar conterminous arcs, which have their chords coincident, have a common tangent.




Let the similar conterminous arcs $APB$, $Apb$, have their chords $AB$, $Ab$ coincident, and let $APp$, $AQq$ be any other coincident chords; then since the curves are similar,

$$AQ : Aq :: AB : Ab :: AP : Ap.$$

Hence the arcs $AQP$, $Aqp$ are similar, and therefore, if $P$ move up to $A$, the arcs $AP$, $Ap$ being always similar will vanish together, and the chord $APp$ in its ultimate position will be a tangent to both.

* [Newton's demonstration of this Lemma is as follows:

Nam si angulus ille non evanescit, continebit arcus $ACB$ cum tangente $AD$ angulum rectilineo equalem, et propterea curvatura ad punctum $A$ non erit continua; contra hypothesin.

The demonstration given in the text differs from the above chiefly in exhibiting more simply and clearly the idea of *continuity*: if the curvature of a curve at any point is *continued*, then the angle between the tangents at two points indefinitely near together is itself indefinitely small; but if in its ultimate position the line $AB$, which will then be coincident in direction with $AD$, makes with the arc $ACB$ a finite angle, then it must follow that there is a finite angle between the tangents at two points indefinitely near together, that is, there is a *discontinuity* of the curvature.]

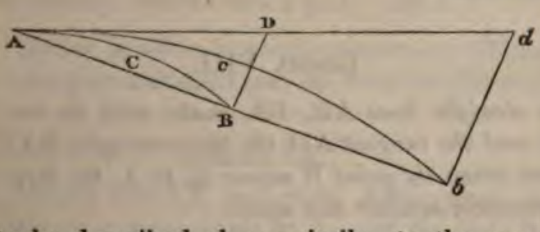
Def. The *subtense* of an arc is a straight line, drawn from one extremity of the arc to meet, at a finite angle, the tangent to the arc at its other extremity.

Obs. The following three Lemmas involve a common principle, which it may be well to endeavour to explain. The purpose of each Lemma is to discover the ultimate value of a ratio, both terms of which become in the limit evanescent, and the difficulty consists in determining this value geometrically. The artifice made use of by Newton is this; he substitutes for the ratio, the ultimate value of which is to be determined, another ratio, which is such, that it is always equal to the given ratio, but yet that its terms become finite and not evanescent in the limit; the difficulty therefore, just now alluded to, does not enter into the determination of the ultimate value of this subsidiary ratio, which being found, the ultimate value of the given ratio is also known, being equal to it.]

## Lemma VII.

*If* BD *be a subtense of the arc* ACB *of continued curvature, and* B *move up to* A, *then will the ultimate ratio of the arc* ACB, *the chord* AB, *and the tangent* AD *be a ratio of equality.*

Let $AD$ be produced to some fixed* point $d$, and as $B$ moves up to $A$, suppose $db$ always drawn through $d$ parallel to $DB$ to meet $AB$ produced in $b$. Also on $Ab$ suppose an




arc $Acb$ to be described always similar to the arc $ACB$, and having therefore $ADd$ for its tangent.

* [For simplicity's sake the point $d$ is spoken of as a *fixed* point, but this condition is not necessary to the proof: the only necessary condition is, that $Ad$ should always be *finite*. A similar observation applies to the next two Lemmas.]

Then, by similar figures, we shall always have,

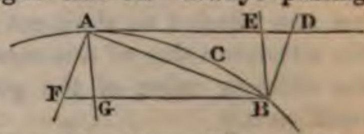
$$AB : ACB : AD :: Ab : Acb : Ad;$$

and since this proportion is always true, it is true in the limit when $B$ has moved up to $A$.

But, in this case, the angle $bAd$ vanishes, and therefore the point $b$ coincides with $d$, and the lines $Ab$, $Ad$, and therefore $Acb$ which lies between them, are equal.

Hence also the arc $ACB$, the chord $AB$, and the tangent $AD$, which are always in the same proportion as $Acb$, $Ab$, and $Ad$, are ultimately equal. Q. E. D.

Cor. 1. Hence if through $B$, $BF$ be drawn parallel to the tangent, cutting any straight line $AF$ always passing through $A$, this line $BF$ will ultimately have a ratio of equality to the evanescent arc $ACB$, because if we complete the parallelogram $AFBD$ it has always a ratio of equality to $AD$.




Cor. 2. And if through $B$ and $A$ any number of straight lines $BE$, $BD$, $AF$, $AG$, be drawn, cutting the tangent $AD$ and the line $BF$ which is parallel to it; the ultimate ratio of the lines $AD$, $AE$, $BF$, $BG$, and the chord and arc $AB$ will be a ratio of equality.

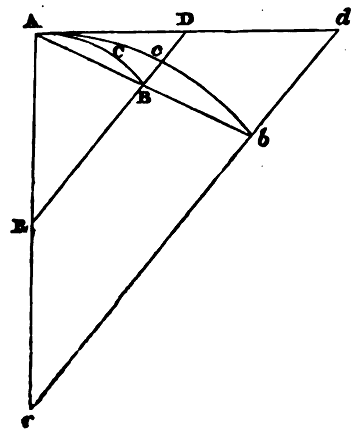
Cor. 3. Hence in all reasonings concerning ultimate ratios, the arc, chord, and tangent may be used indifferently one for another.

## Lemma VIII.

*If two straight lines* AR, BR, *make with the arc* ACB *the chord* AB, *and the tangent* AD, *the three triangles* RAB, RACB, RAD; *then when the point* B *moves up to* A, *the three triangles will be ultimately similar and equal.*

Let $AD$ be produced to some fixed point $d$, and as $B$ moves up to $A$ suppose $dbr$ always drawn through $d$ parallel to $DBR$ to meet $AB$ produced in $b$, and $AR$ produced in $r$. Also, on $Ab$ suppose an arc $Acb$ to be described always

similar to the arc $ACB$, and having therefore $ADd$ for its tangent.



Then, by similar figures, we shall always have,

$$RAB : RACB : RAD :: rAb : rAcb : rAd.$$

And since this proportion is always true, it is true in the limit when $B$ has moved up to $A$.

But, in this case, the angle $bAd$ vanishes, and therefore the point $b$ coincides with $d$, and $Ab$ with $Ad$; and the triangles $rAb$, $rAd$, and therefore $rAcb$ which is intermediate to them, are similar and equal.

Hence also, the triangles $RAB$, $RACB$, $RAD$ which are always similar to, and in the same proportion as $rAb$, $rAcb$, $rAd$, are ultimately similar and equal. Q. E. D.

COR. Hence in all reasonings concerning ultimate ratios, the three triangles aforesaid may be used indifferently for one another.

### LEMMA IX.

*If the straight line* AE *and curve* ABC, *given in position, cut each other in a finite angle at* A, *and the lines* BD, CE *be drawn, meeting the line* AE *in any other finite angle, and the curve in* B *and* C; *then, if the points* B *and* C *move up to* A, *the curvilinear triangles* ABD, ACE *will be ultimately in the duplicate ratio of their sides.*

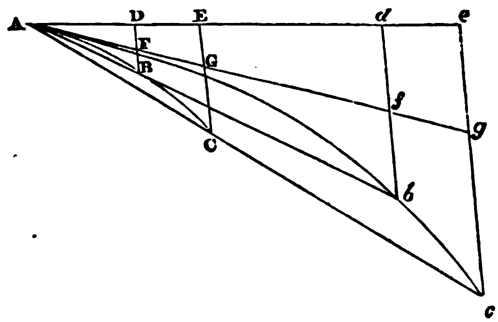
Let $AE$ be produced to some fixed point $e$, and take $Ad$ such that

$$Ad : Ae :: AD : AE.$$

Draw $db$, $ec$ parallel to $DB$, $EC$, to meet $AB$, $AC$ produced in $b$ and $c$. On $Ac$ describe an arc of a curve similar to $ABC$, which will pass through $b$, because by similar figures

$$Ab : Ac :: AB : AC.$$

As the points $B$, $C$ move up to $A$, suppose the curve $Abc$ to change its form so as to be always similar to the curve $ABC$; then the area $ABD$ will always be similar to $Abd$, and $ACE$ to $Ace$; hence




$$\text{area } ABD : \text{area } Abd :: AD^2 : Ad^2,$$

$$\text{and area } ACE : \text{area } Ace :: AE^2 : Ae^2;$$

$$\text{but } AD^2 : AE^2 :: Ad^2 : Ae^2;$$

$$\therefore \text{area } ABD : \text{area } ACE :: \text{area } Abd : \text{area } Ace,$$

and this, being true always, will be true ultimately when $B$ and $C$ have moved up to $A$.

But, in this case, if $AFGfg$ be the common tangent to the two arcs at $A$, the angles $bAf$, $cAg$ will vanish, and the areas $Abd$, $Ace$ will be ultimately the areas $Afd$, $Age$; but

$$\text{area } Afd : \text{area } Age :: Ad^2 : Ae^2,$$
$$:: AD^2 : AE^2;$$

$\therefore$ also, ultimately,

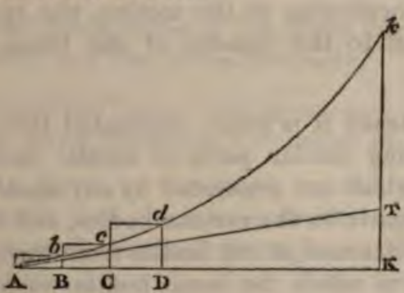
$$\text{area } ABD : \text{area } ACE :: AD^2 : AE^2.$$

Q.E.D.

## Lemma X.

*The spaces, described from rest by a body under the action of any finite force, are in the beginning of the motion as the squares of the times in which they are described*.*

Let time be represented by spaces set off along the line $AK$, and velocity generated by lines perpendicular to $AK$. And let the time be divided into a number of equal intervals $AB$, $BC$, $CD$, ...... ; let $Bb$, $Cc$, $Dd$, ...... be the velocities acquired in the times $AB$, $AC$, $AD$ ...... ; and complete the parallelograms $Ab$, $Bc$, $Cd$......




Suppose the force to act by impulses, which would cause the body to move during the times $AB$, $BC$, $CD$, ...... uniformly, with the velocities $Bb$, $Cc$, $Dd$, ...... respectively; then the spaces described during the 1st, 2nd, 3rd ...... intervals will be represented by the parallelograms $Ab$, $Bc$, $Cd$, ...... and the space described in any given time ($AK$) by the sum of such parallelograms. But, if we suppose the intervals of time indefinitely decreased in magnitude and increased in number, the series of impulses will constitute a continuous force, and the sum of the parallelograms will (by Lemma II.) be equal to the area $AKk$.

* [The following is the original Latin of the demonstration, which for clearness' sake has been given at greater length in the text. The letters refer to the figure of Lemma IX.

Exponantur tempora per lineas $AD$, $AE$, et velocitates genitæ per ordinatas $DB$, $EC$; et spatia his velocitatibus descripta, erunt ut areæ $ABD$, $ACE$ his ordinatis descriptæ, hoc est, ipso motus initio (per lemma IX.) in duplicatâ ratione temporum $AD$, $AE$. Q.E.D.]

Hence, if a finite force act during any times $AD$ and $AK$, we shall have,

space in time $AD$ : space in time $AK$ :: area $ADd$ : area $AKk$.

Also the angle at which the curve $Ak$, or the tangent $AT$, meets the line $AK$ is finite, for since the force is finite the ratio $Kk : AK$ is always finite, and therefore the ratio $KT : AK$ (to which the ratio $Kk : AK$ is ultimately equal) is finite.

Hence, (by Lemma IX.) ultimately,

$$\text{area } ADd : \text{area } AKk :: AD^2 : AK^2;$$

that is, in the beginning of the motion, the spaces described are proportional to the squares of the times of describing them*. Q.E.D.

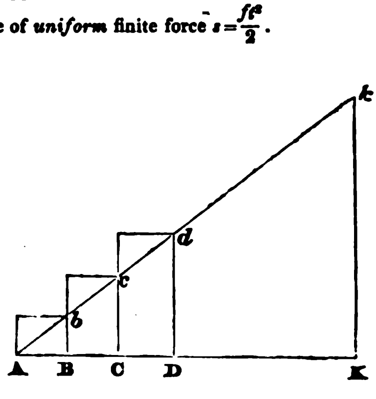
COR. 1. Hence it is easily concluded that the errors of bodies describing similar parts of similar figures in proportionate times, which are generated by any equal forces applied in a similar manner to the various bodies, and which are measured by the distances of the bodies from those places in the similar figures to which the same bodies would come in the same proportionate times without the action of those forces, are approximately as the squares of the times in which they are generated.

* [The same mode of demonstration is applicable to the proposition already proved, (page 291, Art. 29), namely, that in the case of *uniform* finite force $s = \frac{ft^2}{2}$.

For, let time be represented by spaces set off along the line $AK$, and velocity generated by lines perpendicular to it, as before. Then, since the velocity is proportional to the time in which it is generated, the points $b$, $c$, $d$......will be in a straight line; and the space described in the time $AK$ will be represented by the triangle $AKk$, or by half the rectangle under $AK$ and $Kk$;

$$\therefore \text{space} = \frac{\text{vel.} \times \text{time}}{2} = \frac{\text{force} \times \overline{\text{time}}|^2}{2},$$

$$\text{or } s = \frac{ft^2}{2}.]$$



Cor. 2. Also the errors which are generated by unequal forces, similarly applied to similar parts of similar figures, are as the forces and the squares of the times conjointly.

Cor. 3. The same thing is true of the spaces which bodies describe under the action of different forces. These are in the beginning of the motion conjointly as the forces and the squares of the times.

Cor. 4. And therefore in the beginning of the motion the forces are as the spaces described directly, and the squares of the times inversely.

Cor. 5. And the squares of the times are as the spaces described directly, and the forces inversely.

[If $F$ represent the force, $S$ the space, and $T$ the time, we may deduce the three preceding corollaries as follows.

Accelerating force is measured by the velocity which would be generated in a given time divided by the time, the force being supposed uniform throughout the time. (See Dynamics, page 274, Art. 14.) Now, if the force were to be uniform and of the same intensity as at $A$, the curve $Ak$ would coincide with the tangent $AT$;

$$\therefore F = \frac{KT}{AK} = \frac{KT \,.\, AK}{AK^2} = 2\,\frac{\text{triangle } AKT}{AK^2}$$

$$= 2 \text{ limit } \frac{\text{area } AKk}{AK^2}$$

$$= 2 \text{ limit } \frac{S}{T^2}.$$

And the effect produced by $F$ upon the body is independent of any motion, which it may have when $F$ begins to act upon it. Hence, if $S$ be the space through which a force $F$ draws a body, in the time $T$, from the position which it would have occupied if $F$ had not acted, $F = 2 \text{ limit } \dfrac{S}{T^2}$.]

## SCHOLIUM.

If quantities of different kinds be compared one with another, and any one of them be said to be directly or inversely as another; the meaning is, that the former is increased or diminished in the same ratio as the latter or as its reciprocal. And if any one of them be said to be as any other two or more directly or inversely, the meaning is, that the first is increased or diminished in the ratio which is compounded of the ratios in which the others or their reciprocals are increased or diminished. As for instance, if $A$ should be said to be as $B$ directly and $C$ directly and $D$ inversely; the meaning is, that $A$ is increased or diminished in the same ratio as $B \times C \times \frac{1}{D}$, that is, that $A$ and $\frac{BC}{D}$ are to each other in the given ratio.

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## [DIGRESSION CONCERNING THE CURVATURE OF CURVE LINES.

1. *On the measure of the curvature of a curve at any point.*

Let $PQ$, $Pq$ be two curves having the same tangent at $P$, then the curvatures of these two curves at the point $P$ will be compared, by comparing the rate at which their deflection from the common tangent *begins* to take place. Draw the subtense $TQq$, and join $PQ$, $Pq$; then if $TQq$ were to move parallel to itself up to $P$, $PQ$, $Pq$ would ultimately become tangents to the curves $PQ$, $Pq$ respectively, and the ultimate value of the ratio of the angles $TPQ$, $TPq$ will therefore measure the ratio of the curvatures of the curves at $P$.

$$\therefore \frac{\text{curvature of } PQ \text{ at } P}{\text{curvature of } Pq \text{ at } P} = \text{limit}\,\frac{QPT}{qPT} = \text{limit}\,\frac{\sin QPT}{\sin qPT}$$

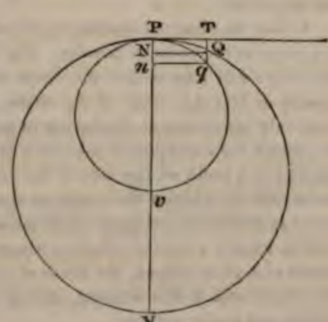
$$= \text{limit}\,\frac{\dfrac{QT}{PQ}\sin PTQ}{\dfrac{qT}{Pq}\sin PTq} = \text{limit}\,\frac{QT}{qT}.$$

2. Another, and perhaps a simpler, way of viewing this proposition is to consider, that if from any point $T$ in the tangent we draw a perpendicular to meet the two curves in $Q$ and $q$ respectively, the deflections from the tangent will be measured by the distances of $Q$ and $q$ from the tangent, that is, by $QT$ and $qT$. Hence the ratio of the deflections of the two curves from the tangent, in the immediate neighbourhood of $P$, will be measured by the ultimate value of the ratio $\frac{QT}{qT}$. We have supposed here that $QT$ and $qT$ are drawn perpendicular to the tangent, but the ratio will be the same in the limit at whatever angle they are drawn.

3. The curvature of a circle is the same throughout and depends only on the radius, as we shall shew immediately; hence it is convenient to speak of the curvature of a curve at a proposed point, as being the same as that of a circle of given radius.

4. *The curvatures of two circles are to each other in the inverse ratio of their diameters.*

Let $PQV$, $Pqv$ be two circles having diameters $PV$, $Pv$, and a common tangent $PT$; from any point $T$ in the tangent draw $TQq$ parallel to $PV$, and draw the ordinates $QN$, $qn$.




Then $\dfrac{\text{curvature of } PQV}{\text{curvature of } Pqv}$

$$= \text{limit}\,\frac{QT}{qT} = \text{limit}\,\frac{PN}{Pn}$$

$$= \text{limit} \frac{\frac{NQ^2}{NV}}{\frac{nq^2}{nv}} = \text{limit} \frac{nv}{NV} = \frac{Pv}{PV}.$$

5. Hence, if at any point of a curve we draw a circle having the same tangent and curvature as the curve has at that point, we may take the reciprocal of its diameter as the measure of the curvature of the curve at that point, and the curvature is said to be finite when the diameter of the circle is finite.

This circle is called *the circle of curvature**, and the radius, diameter, and chord of the circle, are called respectively the *radius, diameter*, and *chord of curvature*.

We shall now shew how to calculate the chord and radius of curvature of a curve, and apply the method to the Conic Sections.

* A convenient mode of viewing the circle of curvature is to consider it as a circle drawn through three points in the curve which are indefinitely near together.

Suppose $PQR$ to be three points in a curve indefinitely near together, then we know, by Euclid, IV. 5, how to describe a circle about the triangle $PQR$. Bisect $PQ$, $QR$ in $a$, $b$; draw $aO$, $bO$ perpendicular to $PQ$ and $QR$, to intersect in $O$, then will $O$ be the centre of the circle of curvature.

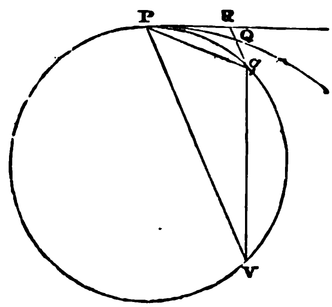

It is easy to prove, if necessary, that the circle here defined is really identical with the circle of curvature as defined in the text; for since the circle is described through the three points $P$, $Q$, $R$ of the curve, the angle contained between the lines $PQ$, $QR$ may be considered to be the angle between the arc and the tangent either in the circle or the curve, that is, the angle between the arc and the tangent is the same for the circle and the curve, in other words the curvature is the same.

Also it is easy from this definition of the circle of curvature to deduce the expression for its radius.

It may be observed concerning the circle of curvature, that it generally passes through the curve at the point of contact. For it is manifest that in leaving the point of contact the curve will pass within the circle or without it, according as its curvature becomes greater or less than that of the circle; now in general the curvature of a curve is continuously increasing or decreasing in passing from point to point, hence since at the point of contact the curvature of the curve is the same as that of the circle, it will be less in passing to a point on one side of the point of contact and greater on the other, that is, the curve will lie without the circle on one side, and within on the other. At points of a curve at which the curvature after increasing begins to decrease, or the reverse, the circle will be wholly within or wholly without the curve. For example, at the extremity of the major axis of an ellipse, the circle of curvature lies within the ellipse, at the extremity of the minor axis it lies without, and at other points it crosses the curve, and lies partly within and partly without.

6. *If* PqV *be the circle of curvature at any point* P *of a curve* PQ, *and* PV *a chord of the circle drawn in any given direction, then*

$$PV = \text{limit}\,\frac{\text{arc}^2}{\text{subtense parallel to the chord}}.$$




Let $RQq$ be the subtense; join $Pq$, $qV$. Then the triangles $PVq$, $PRq$ are evidently similar;

$$\therefore PV = \frac{Pq^2}{Rq} = \text{limit}\,\frac{Pq^2}{Rq} = \text{limit}\,\frac{PQ^2}{RQ},$$

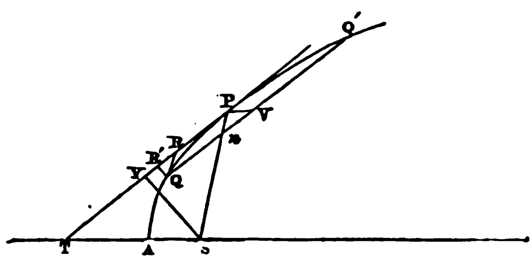
since, by hypothesis, limit $\dfrac{RQ}{Rq} = 1$.

COR. Hence,

$$\text{diameter of curvature} = \text{limit}\,\frac{\text{arc}^2}{\text{subtense perpendicular to tangent}}.$$

7. *To find the chord of curvature through the focus, and the diameter of curvature, at any point of a parabola.*

Draw the tangent $PT$, $QR$ parallel to $SP$, $QVQ'$ parallel to $PT$, and $PV$ parallel to the axis. And let $SP$, $QQ'$ intersect in $n$, then $Pn = PV$. (Prop. II. page 164.)




$$\text{Chord of curvature through } S = \text{limit}\,\frac{PQ^2}{RQ} = \text{limit}\,\frac{Qn^2}{Pn}$$

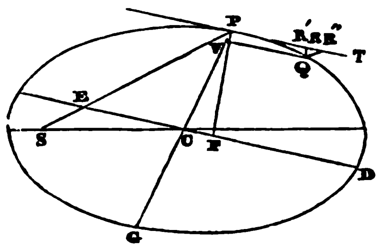
$$= \text{limit}\,\frac{QV^2}{PV} = 4SP. \quad \text{(Prop. IX. page 169.)}$$

To find the diameter of curvature, draw $QR'$, $SY$, perpendicular to the tangent: then,

$$\text{diameter of curvature} = \text{limit}\,\frac{PQ^2}{R'Q} = \text{limit}\,\frac{PQ^2}{RQ}\,\frac{1}{\sin SPY}$$

$$= \frac{4SP}{\sin SPY} = 4\,\frac{SP^2}{SY}.$$

8. *To find the chord of curvature through the centre, the diameter of curvature, and the chord of curvature through the focus, at any point of an ellipse.*




Let $C$ be the centre; draw $QR$ parallel to $CP$, $QV$ to the tangent $PT$: then, chord through the centre

$$= \text{limit}\,\frac{PQ^2}{RQ} = \text{limit}\,\frac{QV^2}{PV} = \text{limit}\,\frac{CD^2}{CP^2}\cdot VG \text{ (Prop. VIII. page 181)}$$

$$= 2\,\frac{CD^2}{CP}, \text{ since } VG = 2CP \text{ ultimately.}$$

Draw $PF$ perpendicular to $CD$, and $QR'$ to $PT$; then,

$$\text{diameter of curvature} = \text{limit}\,\frac{PQ^2}{R'Q} = \text{limit}\,\frac{PQ^2}{RQ}\,\frac{1}{\sin QRR'}$$

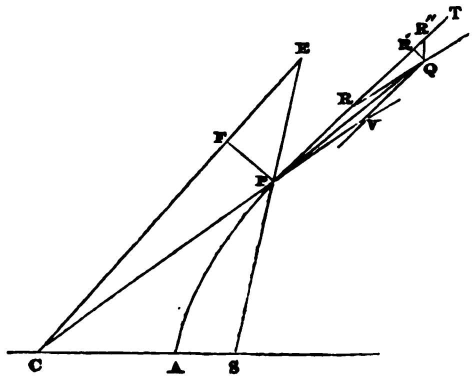
$$= 2\,\frac{CD^2}{CP}\,\frac{1}{\sin PCD} = 2\,\frac{CD^2}{PF}.$$

Again, join $SP$, cutting the conjugate in $E$, and draw $QR''$ parallel to it; then, chord of curvature through the focus

$$= \text{limit}\,\frac{PQ^2}{QR''} = \text{limit}\,\frac{PQ^2}{QR'}\sin QR''R'$$

$$= 2\,\frac{CD^2}{PF}\sin PEF = 2\,\frac{CD^2}{PE} = 2\,\frac{CD^2}{AC} \text{ (Prop. III. Cor. page 174.)}$$

9. *The same for the hyperbola.*

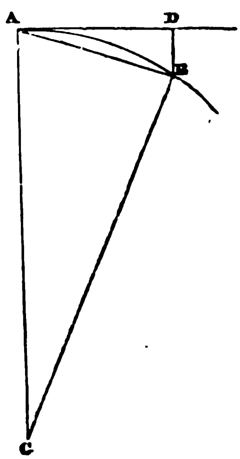



The investigations are the same as for the ellipse; we shall however subjoin a figure.

Obs. The three preceding propositions belong properly to the treatise on Conic Sections, but could not be introduced until the student was familiar with the principles of limits.

10. The following proposition will be required in the succeeding Lemma.

If in the curve $AB$, $AG$, $BG$ be drawn perpendicular to the tangent $AD$ and chord $AB$ respectively; then, when $B$ moves up to $A$, $AG$ will be ultimately the diameter of curvature at $A$.




Draw $BD$ parallel to $AG$, then the triangles $GAB$, $ABD$ are similar;

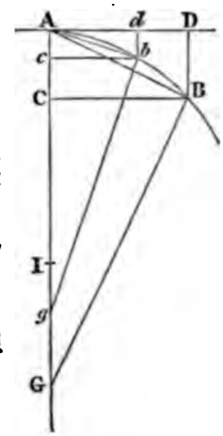
$$\therefore AG = \frac{AB^2}{BD};$$

$$\therefore \text{limit } AG = \text{limit } \frac{AB^2}{BD} = \text{limit } \frac{(\text{arc } AB)^2}{BD}$$

$$= \text{diam. of curvature.]}$$

## Lemma XI.

*In curves of finite curvature, the subtenses are ultimately in the ratio of the squares of the chords of conterminous arcs.*




Let $AbB$ be the curve, having the curvature at $A$ finite.

Case 1. Let the subtenses $bd$, $BD$, be perpendicular to the tangent. Draw $bg$, $BG$ perpendicular to the chords $Ab$, $AB$, and let them meet the normal at $A$ in $g$ and $G$ respectively.

Then, when $b$ and $B$ move up to $A$, $g$ and $G$ will ultimately coincide with $I$ the extremity of the diameter of curvature.

By similar triangles, $BAD$, $AGB$, and $bAd$, $Agb$,

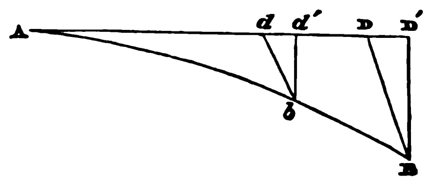
$$BD = \frac{AB^2}{AG},\ bd = \frac{Ab^2}{Ag};$$

$$\therefore\ BD : bd :: \frac{AB^2}{AG} : \frac{Ab}{Ag}.$$

$$\text{And, ultimately, } BD : bd :: \frac{AB^2}{AI} : \frac{Ab^2}{AI},$$

$$:: AB^2 : Ab^2.$$

Case 2. Let the subtenses be inclined at any given angle to the tangent. Draw $bd'$, $BD'$, perpendicular to the tangent;




then, by similar triangles, $BDD'$, $bdd'$,

$$BD : bd :: BD' : bd';$$

but, ultimately, $BD' : bd' :: AB^2 : Ab^2$, by Case 1;

$$\therefore \text{ ultimately, } BD : bd :: AB^2 : Ab^2.$$

CASE 3. Suppose the angle $D$ not to be given, but let the lines $BD$, $bd$ pass through a fixed point, or let $B$ and $b$ approach $A$ according to any other fixed law.

Then, since the angles $D$ and $d$ are formed according to a common law, they will continually approximate to each other as $B$ and $b$ approach $A$, and will be ultimately equal. Hence this case is reduced to the preceding, and the Lemma is therefore still true. Q.E.D.

COR. 1. Hence when the tangents $AD$, $Ad$, the arcs $AB$, $Ab$, and the sines $BC$, $bc$ become ultimately equal to the chords $AB$, $Ab$; their squares will also ultimately be as the subtenses $BD$, $bd$.

COR. 2. The squares of the same lines are also ultimately as the squares of the sagittæ* of the arcs, which bisect the chords and converge to a given point. For those sagittæ are as the subtenses $BD$, $bd$.

COR. 3. And therefore the sagitta is in the duplicate ratio of the time in which a body describes the arc with a given velocity.

COR. 4. The rectilinear triangles $ADB$, $Adb$ are ultimately in the triplicate ratio of the sides $AD$, $Ad$, or as $AD^3 : Ad^3$, and in the sesquiplicate ratio of the sides $DB$, $db$, or as $DB^{\frac{3}{2}} : db^{\frac{3}{2}}$; they being in the ratio compounded of the ratio of $AD$ to $Ad$, and $DB$ to $db$. So also the triangles $ABC$, $Abc$ are ultimately in the triplicate ratio of the sides $BC$, $bc$.

COR. 5. And since $DB$, $db$, are ultimately parallel and in the duplicate ratio of $AD$ to $Ad$, the curvilinear areas $ADB$, $Adb$ will be ultimately (by the nature of the parabola†) equal to two-thirds of the rectilinear areas $ADB$, $Adb$; and the

* [The sagitta of an arc is a line drawn from a point in the chord to a point in the arc.]

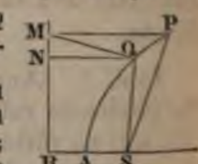
† [The arc $AB$ in all curves of finite curvature may ultimately be taken as the arc of a parabola, having $A$ for its vertex and $AI$ for its axis. For it appears from Cor. 1, that ultimately $BC^2 \propto AC$, which is the property of the parabola. Conics, Prop. v. page 167.

segments $AB$, $Ab$ will be the third parts of the same triangles. And hence these areas and these segments will be in the triplicate ratio of the tangents $AD$, $Ad$, or of the chords or arcs $AB$, $Ab$.

## SCHOLIUM.

In all these propositions we suppose the angle of contact to be neither infinitely greater nor infinitely less than the angles of contact which circles have with their tangents; that is, we suppose the curvature at $A$ to be neither infinitely small nor infinitely great, or $AI$ to be finite. For $DB$ might be taken proportional to $AD^3$; in which case no circle could be drawn through $A$ between the tangent $AD$ and the curve $AB$, and the angle of contact will therefore be infinitely less than in the circle. And, in like manner, if $DB$ be taken successively as $AD^4$, $AD^5$, $AD^6$, $AD^7$, &c., a series of angles of contact will be formed which may be continued indefinitely, and of which each will be indefinitely less than the preceding. And if $DB$ be taken successively as $AD^2$, $AD^{\frac{3}{2}}$, $AD^{\frac{4}{3}}$, $AD^{\frac{5}{4}}$, &c., another infinite series of angles of contact will be formed, of which the first will be of the same kind as in the circle, the second infinitely greater, and each successive angle infinitely greater than the preceding. Also between any two of these angles an infinite series of other angles of contact may be

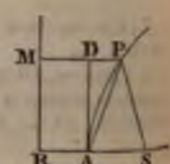
And the property of the parabola which Newton here assumes may be thus proved:




Let $A$ be the vertex of the parabola, $MR$ the directrix, $P$, $Q$ two contiguous points. Draw $PM$, $QN$ perpendicular to the directrix and join $QM$, $SP$, $SQ$.

Then since the tangent at $P$ bisects the angle $MPS$, and $MP = SP$, $MPQ$, $SPQ$ will be ultimately equal triangles; and $MPQ$ ultimately $= MQN$; $\therefore SPQ$ ultimately $= \frac{1}{2}\ MNQP$; and supposing the whole arc $AP$ be cut up into portions such as $PQ$ we have, *componendo*,

$$\text{area } APS = \tfrac{1}{2} \text{ area } ARMP = \tfrac{1}{3} \text{ area } MRSP.$$




This being premised, draw $AD$ a tangent at $A$, then

$$\text{area } AP + APS = \tfrac{1}{3}(MRAD + APS + ADP)$$
$$= \tfrac{1}{3}(3APS + ADP) \quad \text{Euclid, I. 41 \& 36.}$$
$$\therefore \text{area } AP = \tfrac{1}{3} ADP,$$

which is the proposition assumed.]

inserted, of which each is either infinitely less or infinitely greater than the next to it in order. As for instance, if between the terms $AD^2$ and $AD^3$ should be inserted,

$$AD^{\frac{13}{6}},\ AD^{\frac{11}{5}},\ AD^{\frac{9}{4}},\ AD^{\frac{5}{2}},\ AD^{\frac{7}{3}},\ AD^{\frac{8}{3}},\ AD^{\frac{11}{4}},\ AD^{\frac{14}{5}},\ AD^{\frac{17}{6}},\ \&c.$$

And again, between any two angles of this series may be inserted a new series of intermediate angles differing infinitely each from another. Nor is there any limit to this process.

The propositions which have been demonstrated concerning curved lines and the included areas, may easily be applied to the curve surfaces and contents of solids. These lemmas have been premised for the sake of avoiding the tedious methods of the old geometers by the *reductio ad absurdum*. The demonstrations are rendered more brief by the method of *indivisibles;* but since there is some difficulty about the hypothesis of indivisibles, and an apparent want of mathematical exactness, it seemed better to reduce the demonstration of the propositions which follow to the ultimate sums and ratios of evanescent quantities, and to the prime sums and ratios of nascent quantities, that is, to the *limits* of sums and ratios; and to give the demonstrations of those limits as briefly as possible. For the method of *limits* gives the same results as that of *indivisibles**, and the principles having been clearly proved we shall be able to use them with the greater confidence. Wherefore, in what follows, if quantities should ever be spoken of as consisting of particles, or small portions of curves be taken as straight lines, the idea to be entertained is not that of *indivisibles*, but of *evanescent divisible quantities*, not that of sums and ratios of determinate parts, but the *limits* of sums and ratios; and the force of the demonstrations will depend upon their being deducible from the preceding Lemmas.

* [The method of Indivisibles was introduced by Cavaleri, in 1635, in his *Geometria Indivisibilium*. According to this method a line is considered to be made up of indivisible elements or points, so that two lines may be compared by comparing the number of points which they respectively contain; a plane figure is considered to be made up of parallel lines, and a solid of surfaces. This hypothesis, though deficient in philosophical strictness, is nevertheless convenient as a basis for calculations, and will manifestly lead to the same results as Newton's more rigorous method of limits.]

The objection may be made, that evanescent quantities have in reality no ultimate proportion; forasmuch as before they vanish the proportion cannot be said to be ultimate, and after they have vanished there is no proportion at all. But the same argument would prove that a body in arriving at a certain place has no ultimate velocity; because before the body arrives the velocity cannot be said to be ultimate, and after it has arrived there is no velocity at all. And the answer to the objection is simple; namely, that by the *ultimate* velocity is intended that velocity with which the body moves, neither before the body reaches its ultimate position, nor after it has reached it, but at the moment when it reaches it; that is, the velocity with which the body reaches its ultimate position, and with which the motion ceases. And in like manner by the ultimate ratio of evanescent quantities is to be understood the ratio of the quantities, not *before they vanish*, nor *after they have vanished*, but *when they vanish*. So also the prime ratio of nascent quantities is the ratio which they have at their first origin. And the prime and ultimate values of quantities are the values which the quantities have at the commencement or termination of their increase or decrease, as the case may be. There is a limit which the velocity can attain at the end of the motion, but cannot exceed. This is the *ultimate* velocity. And a like description may be given of the limit of all quantities or proportions whether nascent or ultimately evanescent. And since this limit is something certain and definite, the determination of it is a strictly mathematical problem. And in new mathematical investigations all previously established mathematical methods and results may be lawfully used.

It may also be contended, that if the ultimate ratios of evanescent quantities be given, the ultimate magnitudes of the quantities will be given; and so every quantity will consist of indivisible elements, contrary to that which Euclid has proved concerning incommensurables in the tenth book of his elements. But this objection is based on a false hypothesis. Those ultimate ratios with which the quantities vanish are not in reality the ratio of the ultimate quantities, but the limits to which the ratios of the quantities continually

approximate when the quantities are indefinitely diminished, and to which they may be made to approach more nearly than by any assignable difference, but which they never pass, nor reach until the quantities are indefinitely diminished. The thing will be seen more clearly by reference to quantities indefinitely great. If two quantities of which the difference is given be increased indefinitely, their ultimate ratio will be given, namely a ratio of equality; yet the ultimate or greatest quantities of which that is the ratio are not given. In what follows therefore, if at any time for simplicity of conception the phrases *indefinitely small*, or *evanescent*, or *ultimate quantities*, should be used, let it be carefully borne in mind that it is not intended to express quantities of determinate magnitude, but quantities to be diminished without limit.

## SECTION II.

### ON THE METHOD OF FINDING CENTRIPETAL FORCES.

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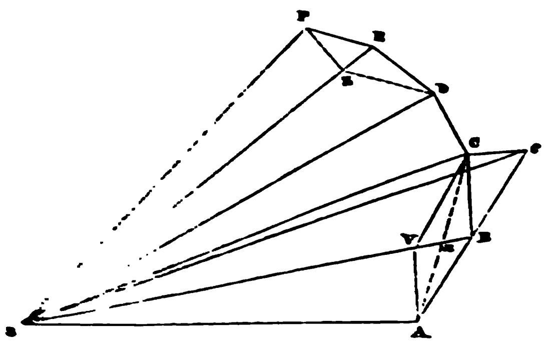
**Prop. I. Theor. I.**

***The areas described by lines drawn from a moving body to a fixed centre of force about which it revolves, are all in one plane, and are proportional to the times of describing them*.***




Let the time be divided into equal parts, and in the first period let the body describe with an uniform velocity the straight line $AB$. In the second period of time, it would, if acted upon by no force, move in $AB$ produced to $c$, $Bc$ being equal to $AB$: and the areas described about the centre $S$ in these two periods, namely, $ASB$, $BSc$ would be equal, being triangles on equal bases and of equal altitude.

But when the body arrives at $B$, suppose it to receive an impulse towards $S$, in consequence of which it moves in the direction $BC$ instead of $Bc$; then, if we draw $cC$ parallel to $BS$, the point $C$ will be the actual place of the body at the

* [The original is as follows, and is here given because it is difficult to render it in a form against which no objection can be made:

Areas, quas corpora in gyros acta radiis ad immobile centrum virium ductis describunt et in planis immobilibus consistere, et esse temporibus proportionales.]

end of the second period; and the area described will be $BSC$, which is in the same plane with $ASB$, because $Bc$ and $BS$ are both in that plane.

Since $BSC$, $BSc$ are triangles on the same base and between the same parallels, $\therefore BSC = BSc = ASB$.

In like manner if impulses tending towards the centre $S$ act at $C, D, E$...causing the body to describe in the successive periods of time the straight lines $CD, DE, EF$...these will all lie in the same plane, and the triangle $SCD$ will be equal to $SBC$, $SDE$ to $SCD$, and $SEF$ to $SDE$.

Therefore equal areas are described in the same plane in equal times; and *componendo* the sums of any number of areas $SADS$, $SAFS$, are to each other as the times of describing them.

Now let the number of the triangles be indefinitely increased and their breadth indefinitely diminished; the perimeter $ADF$ will be ultimately a curve line, and the impulses will become a continuous central force; and the areas $SADS$, $SAFS$ being always proportional to the times of describing them will be so in this case. Hence the areas described, &c. Q.E.D.

Cor. 1. The velocity of a body in a central orbit varies inversely as the perpendicular from the centre on the tangent.

For $AB, BC, CD$,...are ultimately in the direction of the tangents, and proportional to the velocities at the points $A, B, C$...respectively: hence, velocity at $A \propto AB$. But, if we draw a perpendicular $p$ upon $AB$ from $S$, we have

$$\frac{AB \times p}{2} = \text{area } ASB, \text{ which is constant;}$$

$$\therefore AB \propto \frac{1}{p};$$

$$\text{or, velocity at } A \propto \frac{1}{p}.$$

[We may express this otherwise; let $h$ = twice the area described in a unit of time, and let $AB$ be described in the time $t$;

$$\therefore \text{ area } ASB = \frac{ht}{2}, \text{ and } AB = vt;$$

$$\therefore vp = h, \text{ or } v = \frac{h}{p}.]$$

COR. 2. If on $AB$, $BC$, the chords of two arcs described in equal times, we construct the parallelogram $ABCV$, the diagonal $BV$ will, when the arcs are indefinitely diminished, ultimately, if produced, pass through the centre of force.

COR. 3. The intensity of the central force at $B$ is proportional to the line $BV$; that is, if $B'V'$ be the line corresponding to $BV$ at some other point $B'$ of the orbit, then, ultimately,

$$\text{force at } B : \text{force at } B' :: BV : B'V'.$$

COR. 4. The forces by which bodies are drawn from their rectilinear motion in curved paths, are proportional to those sagittæ of arcs described in equal times, the directions of which pass through the centre of force and bisect the chords when those arcs are indefinitely diminished.

For if we join $AC$, cutting $SB$ in $n$, $Bn$ will be ultimately one of the sagittæ, but $Bn = \frac{1}{2} BV$, and hence, by the preceding corollary, in the same orbit the force ultimately $\propto Bn$. Also, in different orbits, if we take arcs described in equal times, the sagittæ will measure the effects of the central forces in equal times, i.e. the forces will be proportional to the sagittæ.

COR. 5. And therefore these same forces are to the force of gravity, as these sagittæ are to the vertical sagittæ of the parabolic arcs, which a projectile describes in the same time.

COR. 6. It follows from the second Law of Motion that the preceding conclusions are still valid, when the plane in which a body moves, together with the centre of force which is situated in it, instead of being at rest, has a uniform rectilinear motion.

## Prop. II. Theor. II.

*A body, which moves in a plane curve, in such a manner that the areas described by lines drawn from it to a point, which either is fixed or moves uniformly in a straight line, are proportional to the time of describing them, is acted upon by a central force tending to that point.*

Case 1. With the same figure as in last proposition, let $S$ be the point; and suppose that a body unattracted by any force would describe the space $AB$ in a given interval of time.

Produce $AB$ to $c$, and make $Bc = AB$; then, if suffered to proceed, the body would, in a second equal interval, arrive at $c$; but at $B$ suppose a sudden impulse communicated, which causes it to move to $C$, $C$ being such that the triangles $SBC$, $SAB$ are equal.

Join $Cc$, $Sc$; then the triangle $SBC = SAB = SBc$; therefore $Cc$ is parallel to $SB$, and therefore the impulse at $B$ was in the direction of $SB$, or tended to $S$. Similarly, if the body receives at equal intervals of time impulses which make it describe equal triangles in equal times, or a polygonal area proportional to the time, the impulses all tend to $S$.

The same will be true, if we suppose the number of the intervals indefinitely increased and their length diminished, in which case the system of impulses becomes a continuous force, and the polygonal area curvilinear. Hence the proposition is true in the case of a fixed centre.

Case 2. The proposition will also be true in the case of the centre being a point which moves uniformly in a straight line; for it is manifest from the second Law of Motion, that the result will be the same whether we suppose the plane in which the areas are described to be fixed, or whether we suppose that plane together with the revolving body and the centre $S$ to have a uniform rectilinear motion.

Hence a body, &c. Q.E.D.

Cor. 1. In free space, or in non-resisting media, if the areas described about a certain point are not proportional to the times, the force does not act along the line joining the

body with that point; but it deviates from that line towards the direction in which the motion takes place, if the description of areas is accelerated; and towards the opposite direction, if the description is retarded.

Cor. 2. Also in the case of resisting media, if the description of areas is accelerated, the force deviates from the line joining the body with the centre towards the direction in which the motion takes place.

### SCHOLIUM.

A body may be acted upon by a centripetal force composed of several forces. In this case the meaning of the preceding proposition is that the *resultant* of all the forces tends to the centre. Also if any force act continually in a direction perpendicular to the area described, the effect of it will be to change the plane of the body's motion, but the amount of area described will not be increased or diminished; and therefore such a force may be neglected in the composition of the forces acting on the body.

### Prop. III. Theor. III.

*A body which describes areas proportional to the times of describing them about another body which is in motion, is acted upon by a force compounded of a centripetal force tending to that other body, and of the accelerating force which acts upon that other body.*

Call the first body $L$ and the second $T$; and suppose each of the bodies to be acted upon by a new force, which shall be equal in magnitude and opposite in direction to that which acts upon $T$. $L$ will continue to describe about $T$ the same areas as before; but the force which acted on $T$ will now be destroyed by an equal and opposite force, and therefore that body will now either be at rest or will move uniformly in a straight line; and the body $L$, under the influence of the resultant of the two forces which act upon it, will describe about $T$ areas proportional to the times. Therefore, by the

preceding proposition, the resultant of these forces tends to $T$ as a centre*. Q. E. D.

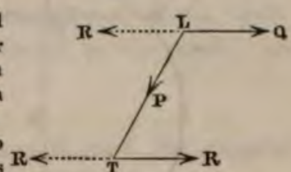
Cor. 1. Hence if a body $L$ describes about $T$ areas proportional to the times, and from the whole force which acts upon $L$ we take away the whole accelerating force which acts upon $T$, the whole remaining force acting upon $L$ tends to $T$ as a centre.

Cor. 2. And if the areas are very nearly proportional to the times, the said force tends very nearly to $T$.

Cor. 3. And *vice versâ*, if the force tends very nearly to $T$, the areas are very nearly proportional to the times.

Cor. 4. If the body $L$ describes areas about $T$, which are very far from being proportional to the times; and if $T$ is either at rest or moving uniformly in a straight line; then, either there is no force upon $L$ tending to $T$, or it is compounded with other much more powerful forces, and the whole resultant force is directed towards some other centre. The same thing holds when the body $T$ moves in any manner whatever, if by the centripetal force upon $L$ tending towards $T$ we understand that which remains after subtracting from it the force acting on $T$.

* [The proof of this proposition seems to be capable of exhibition in its simplest form as follows:




Let the whole force acting upon the body $L$ be resolved into two, one in the direction $LT$ which call $P$, the other parallel to the direction of the force acting upon $T$ which call $Q$. This can always be done. Also let the force on $T$ be called $R$, to which by hypothesis $Q$ is parallel.

Now let a force equal to $R$ and opposite in direction to the force acting on $T$ be applied to both $T$ and $L$. This is represented in the figure by arrows with dotted lines. Then $T$ will on this supposition be at rest or will move uniformly in a straight line, and $L$ will still describe areas uniformly about it, since the same force applied to each of two bodies cannot affect their relative motion. Hence the three forces $R$, $Q$, $P$ acting on $L$ are equivalent to a central force tending towards $T$, (Prop. II.); but the part $P$ is central, and $Q$ and $R$ act in the same straight line, hence it is evident that the resultant of the three cannot be towards $T$ unless $Q=R$.

Hence the forces on $L$ will be $P$ towards $T$, and $R$ parallel to the force acting on $T$, or the single force acting on $L$ will be the resultant of these; in other words, *A body which describes areas, &c.* Q.E.D.

## SCHOLIUM.

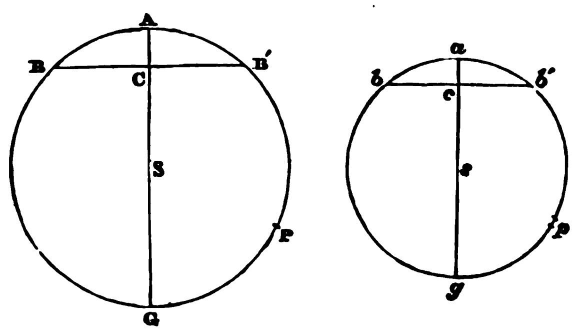
Since the equable description of areas points out the centre, towards which that force tends by which a body is principally affected, and by which it is retained in its orbit, the equability of description of areas forms a convenient method of detecting the centre about which all curvilinear motion in free space takes place.

## Prop. IV. Theor. IV.

*The centripetal forces of bodies, which describe different circles with uniform velocity, tend to the centres of the circles, and are to each other as the squares of arcs, described in the same time, divided by the radii.*

Sectors of circles are proportional to the arcs on which they stand, and therefore the sectors described by the bodies are proportional to the times of describing them, since the bodies move uniformly. Hence the forces tend to the centres of the circles.

Again, let $BAB'$, $bab'$ be indefinitely small arcs described in equal times; join $BB'$, $bb'$, and draw the diameters $GSCA$, $gsca$, which bisect $BB'$, $bb'$ in $C$ and $c$ respectively. Then, (by Prop. I. Cor. 4) ultimately,




$$\text{force at } A : \text{force at } a :: AC : ac,$$

$$:: \frac{BC^2}{GC} : \frac{bc^2}{gc},$$

$$:: \frac{BB'^2}{GC} : \frac{bb'^2}{gc},$$

$$:: \frac{BAB'^2}{AG} : \frac{bab'^2}{ag};$$

but if $AP$, $ap$ be any two arcs described in equal times, since the bodies move uniformly, we have

$$AP : ap = \text{ultimate value of the ratio } BAB' : bab';$$

$$\therefore \text{ force at } A : \text{force at } a :: \frac{AP^2}{AS} : \frac{ap^2}{as}.$$

Hence the centripetal forces, &c. Q.E.D.

COR. 1. Since the arcs are proportional to the velocities, the forces are proportional to $\frac{(\text{velocity})^2}{\text{radius}}$; or if $V$ be the velocity, $R$ the radius, $F$ the central force, then $F \propto \frac{V^2}{R}$.

COR. 2. Let $P$ be the periodic time, then $2\pi R = VP$, (Dynamics, Art. 4, page 268);

$$\therefore F \propto \frac{R}{P^2}.$$

COR. 3. Hence if in two circular orbits the periodic times are equal, the velocities are proportional to the radii, and the central forces are also proportional to the radii; or $F \propto V \propto R$; and *vice versâ*.

COR. 4. If $P^2 \propto R$, then $V^2$ also $\propto R$, and the central forces in the two orbits are equal; and *vice versâ*.

COR. 5. If $P \propto R$, the velocities in the two orbits are equal, and $F \propto \frac{1}{R}$; and *vice versâ*.

COR. 6. If $P^2 \propto R^3$, $V^2 \propto \frac{1}{R}$, and $F \propto \frac{1}{R^2}$; and *vice versâ*.

Cor. 7. And generally, if $P \propto R^n$, $V \propto \frac{1}{R^{n-1}}$, and

$$F \propto \frac{1}{R^{2n-1}}; \text{ and } \textit{vice versâ}.$$

Cor. 8. All these propositions are true concerning the periodic times, velocities, and forces, with which bodies describe similar portions of similar figures having centres of force similarly situated in them. But in applying the same method of demonstration it is necessary to substitute *uniform description of areas* for *uniform motion*, and the *distances of the bodies from the centres* for the *radii of the circles*.

Cor. 9. It follows also from this proposition, that the arc, which a body moving uniformly in a circle under the action of a given central force describes in a given time, is a mean proportional between the diameter of the circle and the space through which the body would fall under the action of the same force and in the same time.

[For, let $T$ be the time, $S$ the space through which the body would fall, $A$ the arc described in time $T$;

then $A = V \,.\, T$; and $2 \,.\, S = F \,.\, T^2$ (Art. 29, page 291).

$$\therefore A^2 = V^2 \,.\, T^2 = F \,.\, R \,.\, T^2, \text{ by Cor. 1, } = 2R \,.\, S,$$

which proves what was required.]

## SCHOLIUM.

The case supposed in Cor. 6, holds for the heavenly bodies, and the nature of the motion of bodies when the centripetal force varies inversely as the square of the distance will therefore be treated more at length hereafter.

By help of the preceding proposition and its corollaries, a centripetal force may be compared with any known force, such as that of gravity. For if a body revolve in a circle, having for its centre the centre of the earth, under the influence of its own gravity, this gravity is the centripetal force. And we can deduce from the falling of heavy bodies both the time of one revolution, and the arc described in any given time, in virtue of Cor. 9. By propositions of this kind Huygens compared the force of gravity with the centrifugal force of revolving bodies.

[Thus if a body of given weight revolve in a circle in a given time, the centrifugal force may be compared with the weight, as in page 317. Moreover, if we suppose the moon to move in a circle having the centre of the earth for its centre, which is approximately true, we may determine the time of a revolution by observation of falling bodies, that is, by observation of the accelerating force of gravity.

For let $R$ be the distance from the earth of the moon, $r$ the earth's radius; then $g\frac{r^2}{R^2}$ is the accelerating force of the earth on the moon. Therefore the time of the moon's revolution measured in seconds

$$=\frac{2\pi R}{\sqrt{\frac{gr^2}{R^2}\times R}}=\frac{2\pi R^{\frac{3}{2}}}{rg^{\frac{1}{2}}}.$$

$R$ and $r$ can be found by astronomical observation, and must be expressed in feet.]

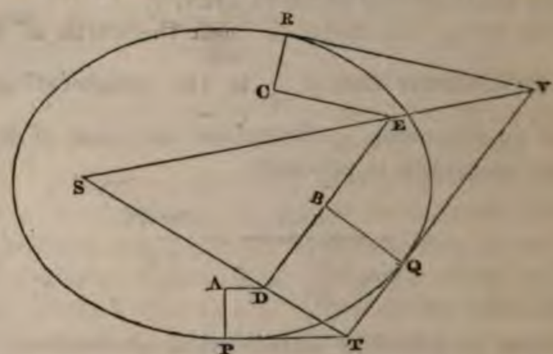
The preceding results may also be obtained in this manner. In any circle let a regular polygon of any number of sides be described. Then if a body moving with a given velocity along the sides of the polygon be reflected from the circle at each of the angles, the force with which the body impinges on the circle at each reflexion will be proportional to the velocity: and therefore the sum of the forces in a given time will be jointly as the velocity and the number of reflexions, that is, if the number of sides of the polygon be given, as the space described in a given time, increased or diminished in the ratio of that space to the radius of the circle; that is, as the square of the space divided by the radius; and therefore if the polygon by the indefinite diminution of its sides be made to coincide with the circle, as the square of an arc described in a given time divided by the radius. This is the *centrifugal force*, with which the body presses the circle; and to this is equal the opposite force, with which the circle continually presses the body towards the centre*.

* [This method of considering centrifugal force has been already treated at greater length in page 315.]

### Prop. V. Prob. I.

*Given the velocity at any three* points of an orbit described by a body under the action of a central force, to find the centre.*

Let the three straight lines $PT$, $TQV$, $VR$ touch the orbit in the points $P$, $Q$, $R$, respectively; and let $S$ be the centre.




Draw the lines $PA$, $QB$, $RC$ perpendicular to these tangents, and make them inversely proportional to the velocities at $P$, $Q$, $R$; i. e. if $V_1$ $V_2$ $V_3$ are the velocities at the three points, make

$$PA : QB : RC :: \frac{1}{V_1} : \frac{1}{V_2} : \frac{1}{V_3}.$$

Through $A$, $B$, $C$ draw the lines $AD$, $DBE$, $CE$ perpendicular to $PA$, $QB$, $RC$. Join $TD$, $VE$: these lines produced will intersect in the centre $S$.

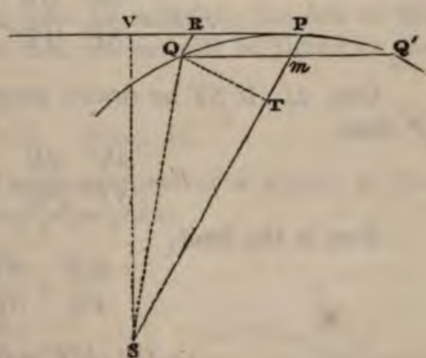
For the perpendiculars from $S$ on the tangents $PT$, $QT$ are inversely proportional to the velocities at $P$ and $Q$, (Prop. I. Cor. 1), and therefore, by construction, directly proportional to $PA$ and $QB$, i.e. to the perpendiculars from $D$ on the tangents; hence $S$, $D$, and $T$ are in the same straight line.

Similarly, it may be shewn, that $S$, $E$, and $V$ are in the same straight line, and therefore the point of intersection of $TD$ and $VE$ produced is the centre $S$.

* [Newton's words are, *datâ quibuscunque in locis velocitate*; the velocity at any three points is sufficient for the solution of the problem.]

## Prop. VI. Theor. V.

*If a body revolves about a fixed centre of force, and a sagitta is drawn to any very small arc, bisecting the chord of the arc and passing through the centre of force, then the force at the middle point of the arc ultimately varies directly as the sagitta and inversely as the square of the time of describing the arc**.




Let $QPQ'$ be the small arc, $S$ the centre of force, $mP$ the sagitta. Draw $PR$ a tangent to the curve at $P$, and $RQ$ parallel to $Pm$; then if $QQ'$ move parallel to itself, since $m$ is the middle point, $QQ'$ will ultimately coincide with the tangent at $P$; therefore $QQ'$ is parallel to $PR$, and therefore

$QR = PM$†.

When the body leaves the point $P$, it would if not acted upon by the central force ($F$) move in the direction $PR$; and if $T$ be the time in which the body moves from $P$ to $Q$, then the space through which it has been drawn by $F$ is $QR$; hence, by Lemma x. Cor. 5,

$$F = 2 \text{ limit } \frac{QR}{T^2} = 2 \text{ limit } \frac{Pm}{T^2} \propto \text{limit } \frac{Pm}{T^2} \ddagger.$$

* [Newton only indicates the proof given in the text; he himself deduces the proposition from Prop. I. Cor. 4, and then adds, "idem facile demonstratur etiam per Cor. 4, Lem. x."]

† [This also appears from regarding $QPQ'$ as an arc of a parabola; $QmQ'$ being a chord to the axis $PS$.]

‡ [This proposition enables us to obtain a measure of the centrifugal force, the nature of which was explained in the treatise on Dynamics, Art. 46, page 313.

For the measure of the centrifugal force is the force necessary at each moment to draw the body from the tangent into the curve, and therefore if we take $QR$ perpendicular to the tangent, we shall have

Cor. 1. Draw $QT$ perpendicular to $SP$, then will

$$F = \frac{2h^2}{SP^2}\frac{QR}{QT^2}, \text{ ultimately.}$$

For the triangle $QSP = \dfrac{SP \,.\, QT}{2}$, ultimately,

$$\therefore SP \,.\, QT = hT, \text{ by Prop. I. Cor. 1};$$

$$\therefore F = \frac{2h^2}{SP^2}\frac{QR}{QT^2}, \text{ ultimately.}$$

Cor. 2. If $SY$ be drawn perpendicular to the tangent at $P$, then

$$F = \frac{2h^2}{SY^2}\frac{QR}{PQ^2}, \text{ ultimately.}$$

For, in the limit,

$$\frac{QT}{PQ} = \frac{SY}{SP};$$

$$\therefore SP^2 \,.\, QT^2 = SY^2 \,.\, PQ^2,$$

$$\text{and } F = \frac{2h^2}{SY^2}\frac{QR}{PQ^2}, \text{ ultimately.}$$

Cor. 3. If $PV$ be the chord of curvature at $P$ through $S$,

$$PV = \text{limit } \frac{PQ^2}{QR};$$

$$\therefore F = \frac{2h^2}{SY^2 . PV}.$$

centrifugal force estimated in the direction of the normal

$$= 2 \text{ limit } \frac{QR}{T^2} = 2 \text{ limit } \frac{QP^2}{T^2} \div \frac{QP^2}{QR}$$

$$= 2 \,\overline{\text{velocity}}\,\rvert^2 \div \text{diameter of curvature, (by Art. 6, Cor. page [illegible])}$$

$$= \frac{V^2}{\rho},$$

where $V$ is the velocity, and $\rho$ the radius of curvature at the given point of the body's path.

Or the same thing would follow from Prop. IV. Cor. 1, by supposing the body to be moving at any moment in the circle of curvature.]

Cor. 4. If $V$ be the velocity at $P$, then by Prop. I. Cor. 1,

$$V = \frac{h}{SY},$$

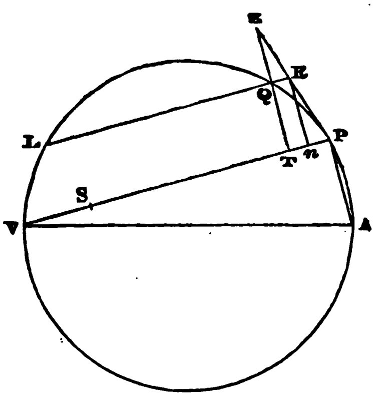
$$\therefore F = \frac{2V^2}{PV}\,*.$$

Cor. 5. Hence, if the form of the orbit in which a body moves be given, we shall be able to calculate the law of the central force. Examples of this process will be found in the following problems.

## Prop. VII. Prob. II.

*A body revolves in the circumference of a circle; to find the law of force tending to any given point.*

Let $VQPA$ be the circle, $S$ the given point, $P$ the position of the body at any given time, $Q$ a point in the orbit very near to $P$.




Through $S$ draw the chord $PSV$, and through $V$ the diameter $VA$; join $PV$; through $Q$ draw $ZQT$ perpendicular to $PV$, and meeting $PZ$ the tangent at $P$ in $Z$; through $Q$ draw $LQR$ parallel to $PV$, and meeting $PZ$ in $R$; and lastly draw $Rn$ parallel to $QT$.

Then $F = \dfrac{2h^2}{SP^2}\dfrac{QR}{QT^2}$, ultimately;

but, (Euclid, III. 36)

$$QR \cdot RL = RP^2;$$

* [This is sometimes expressed by saying, that the velocity in a central orbit is that which would be acquired in falling through one fourth of the chord of curvature, the force being supposed constant.]

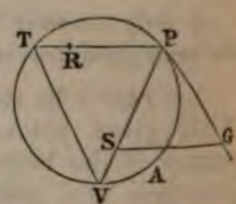
$$\therefore F = \frac{2h^2}{RL \,.\, SP^2}\,\frac{RP^2}{QT^2};$$

$$\text{and } \frac{RP^2}{QT^2} = \frac{RP^2}{Rn^2} = \frac{VA^2}{PV^2} \text{ by similar triangles,}$$

also $RL$ ultimately $= PV$,

$$\therefore F = \frac{2h^2}{SP^2}\,\frac{VA^2}{PV^3} \propto \frac{1}{SP^2 \,.\, PV^3}.$$

COR. 1. If the centre of force is in the circumference, $F \propto \dfrac{1}{SP^5}$.




COR. 2. The force, under the action of which the body $P$ revolves in the circle $APTV$ round the centre of force $S$, is to the force under the action of which the same body $P$ revolves in the same circle and in the same periodic time round any other centre of force $R$, as $SP \,.\, RP^2$ to $SG^3$; where $SG$ is a line drawn from $S$ to the tangent $PG$ and parallel to $RP$.

For by the construction in the preceding proposition, force tending to $S$ : force tending to $R$

$$:: RP^2 \,.\, PT^3 : SP^2 \,.\, PV^3,$$

$$:: SP \,.\, RP^2 : \frac{SP^3 \,.\, PV^3}{PT^3},$$

$$:: SP \,.\, RP^2 : SG^3,$$

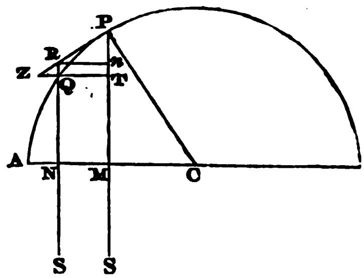
by similar triangles $PSG$, $TPV$.

COR. 3. The force, under the action of which the body $P$ revolves in any orbit round a centre of force $S$, is to the force, under the action of which the same body $P$ revolves in the same orbit and in the same periodic time round any other centre of force $R$, as $SP \,.\, RP^2$ to $SG^3$; where $SG$ is the line drawn from $S$ to the tangent of the orbit, and parallel to $RP$ the distance of $P$ from the other centre $R$.

This follows immediately from the preceding corollary; because we may suppose the body, at any point of its motion, to be moving in the circle of curvature to the orbit at that point.

## Prop. VIII. Prob. III.

*A body describes a semicircle, under the action of a force tending to a centre so distant that the force may be supposed to act in parallel lines; to find the law of force.*




Let $P$ be the position of the body at a given time, $Q$ a contiguous position, $PMS$, $QNS$, the directions of the force at those points; draw the semi-diameter $AC$ cutting those lines at right angles in $M$ and $N$; let $PRZ$ be the tangent at $P$; through $Q$ draw $ZQT$ perpendicular to $PM$ and meeting $PRZ$ in $Z$, and produce $NQ$ to meet $PRZ$ in $R$; join $CP$, and draw $Rn$ perpendicular to $PM$.

Then $F \propto \text{limit}\dfrac{QR}{SP^2 \,.\, QT^2} \propto \text{limit}\dfrac{QR}{QT^2}$, since $SP$ may be considered constant.

But, by Euclid, III. 36,

$$QR\,(QN+RN) = RP^2,$$

$$\text{or, in the limit, } 2\,QR.PM = RP^2,$$

$$\therefore F \propto \text{limit } \frac{RP^2}{2PM.QT^2};$$

$$\text{but } \frac{RP^2}{QT^2} = \frac{RP^2}{Rn^2} = \frac{CP^2}{PM^2} \text{ by similar triangles;}$$

$$\therefore F \propto \frac{CP^2}{2PM^3} \propto \frac{1}{PM^3}.$$

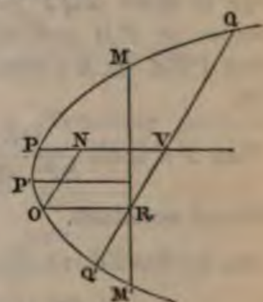
This result may also be immediately deduced from the last proposition; for since $S$ is at an infinite distance, $SP$ may be regarded as constant, and therefore $f \propto \frac{1}{PV^3} \propto \frac{1}{PM^3}$, since $PV = 2PM$.

SCHOLIUM.

By somewhat similar reasoning it may be shewn that a body may move in an ellipse, or a parabola or hyperbola, under the action of a force tending to a centre infinitely distant and varying inversely as the cube of the ordinate drawn towards the centre of force.

[To prove this, however, it will be necessary to make use of the following proposition.

In any conic section if from a point $R$, either within or without the curve, two straight lines be drawn parallel to two given directions, and cutting the curve in $M$, $M'$ and $Q$, $Q'$ respectively, then the ratio $RM \,.\, RM' : RQ \,.\, RQ'$ will be independent of the position of the point $R$.




(1) In the parabola.

Through the point $R$ draw the chord $QRQ'$, and let $V$ be its middle point; draw $PV$, $OR$ parallel to the axis of the parabola, and $ON$ parallel to $QQ'$.

Then $RQ.RQ' = QV^2 - VR^2$ (Euc. II. 5)

$= QV^2 - ON^2 = 4SP.PV - 4SP.PN$ (Prop. IX. p. 169)

$= 4SP.NV = 4SP.OR.$

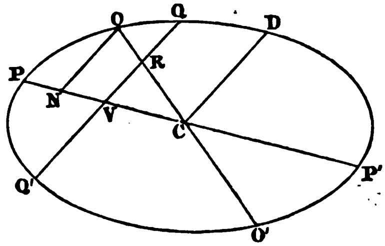
In like manner for any other chord $MRM'$ we should have,

$$RM \,.\, RM' = 4SP'.OR;$$

$$\therefore \frac{RQ \,.\, RQ'}{RM.RM'} = \frac{SP}{SP'}.$$

Now the points $P$ and $P'$ remain the same as long as the chords are drawn parallel to fixed directions: hence the proposition is true.

(2) In the ellipse.




Through the centre $C$ and point $R$ draw the chord $ORCO'$, $CD$ parallel to $QQ'$, $CVP$ conjugate to $CD$, $ON$ parallel to $CD$.

Then

$$RQ.RQ' = QV^2 - VR^2,$$

$$\text{also } RO.RO' = OC^2 - CR^2 = OC^2 - VR^2 . \frac{OC^2}{ON^2}$$

$$= \frac{OC^2}{ON^2}(ON^2 - VR^2);$$

$$\text{but } \frac{QV^2}{CD^2} = \frac{PV.VP'}{CP^2}; \text{ (Prop. VIII. p. 181)}$$

$$\therefore \frac{CD^2 - QV^2}{CD^2} = \frac{CP^2 - PV.VP'}{CP^2} = \frac{CV^2}{CP^2};$$

$$\text{similarly } \frac{CD^2 - ON^2}{CD^2} = \frac{CN^2}{CP^2};$$

$$\therefore \frac{CD^2 - QV^2}{CD^2 - ON^2} = \frac{CV^2}{CN^2} = \frac{VR^2}{ON^2};$$

$$\therefore \frac{CD^2 - QV^2}{VR^2} = \frac{CD^2 - ON^2}{ON^2},$$

$$\frac{CD^2 - QV^2 + VR^2}{VR^2} = \frac{CD}{QN}^2,$$

$$CD^2(ON^2 - VR^2) = ON^2(QV^2 - VR^2),$$

$$\frac{QV^2 - VR^2}{ON^2 - VR^2} = \frac{CD^2}{ON^2};$$

$$\therefore \frac{RQ\,.\,RQ'}{RO\,.\,RO'} = \frac{QV^2 - VR^2}{ON^2 - VR^2}\,\frac{ON^2}{OC^2},$$

$$= \frac{CD^2}{OC^2}.$$

Similarly we should have for any other chord

$$\frac{RM\,.\,RM'}{RO\,.\,RO'} = \frac{CD'^2}{OC^2};$$

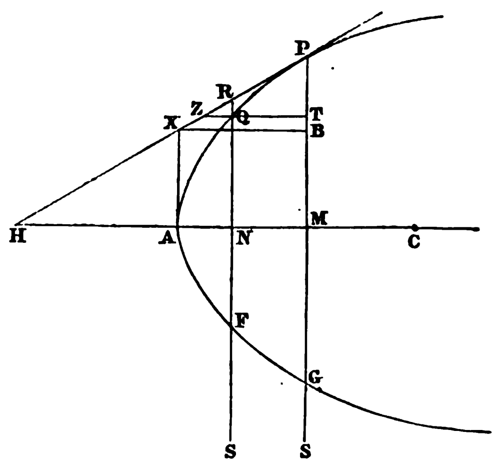
$$\therefore \frac{RQ\,.\,RQ'}{RM\,.\,RM'} = \frac{CD^2}{CD'^2};$$

whence the truth of the proposition is manifest.

A similar demonstration will hold for the hyperbola.

If the point $R$ be exterior to the curve, and the two points $Q$, $Q'$ be indefinitely near together, the chord becomes a tangent, and $RQ\,.\,RQ'$ becomes $RQ^2$, and the proposition is still true.

This being premised, let $PQA$ be a conic section of which the axis is $AC$, and which is described by a body under the action of a force always perpendicular to $AC$; make a construction similar to that for the circle, and let $H$ be the point in which the tangent at $P$ meets the axis, $AX$ the tangent at the vertex, and $XB$ perpendicular to $PM$.




Then $$F \propto \frac{1}{SP^2}\cdot\frac{QR}{QT^2} \propto \frac{QR}{QT^2}.$$

But by the preceding proposition

$$\frac{RP^2}{RQ\,.\,RF} = \frac{XP^2}{XA^2};$$

so by similar triangles,

$$\frac{XP^2}{AM^2} = \frac{RP^2}{QT^2};$$

$$\therefore \frac{XA^2}{AM^2} = \frac{RQ.RF}{QT^2};$$

$$\therefore \frac{QT^2}{QR} = \frac{AM^2.RF}{XA^2} = \frac{2AM^2.PM}{XA^2} \text{ ultimately.}$$

Again from similar triangles,

$$\frac{HM}{PM} = \frac{HA}{XA};$$

$$\therefore F \propto \frac{QR}{QT^2} \propto \frac{HA^2.PM}{AM^2.HM^2} \text{ in all the conic sections.}$$

Now in the parabola, $HA = AM$; therefore

$$HM \propto AM \propto PM^2.$$

$$\therefore F \propto \frac{1}{PM^5}.$$

In the ellipse and hyperbola let $A'$ be the other vertex, and let $C$ be the centre; then

$$\frac{PM^2}{AM.A'M} = \frac{L}{2AC};$$

$$\therefore AM = \frac{2PM^2.AC}{L.A'M};$$

$$\therefore \frac{AM^2.HM^2}{HA^2.PM} = \frac{HM^2}{HA^2}\cdot\frac{4PM^3.AC^2}{L^2.A'M^2} \propto \frac{HM^2.PM^3}{HA^2.A'M^2}.$$

But $CM.CH = AC^2$, (Conics, Prop. v. p. 177, and Prop. v. p. 191)

$$\text{or } \frac{CM}{AC} = \frac{AC}{CH};$$

$$\therefore \frac{CM}{AM} = \frac{CM}{AC - CM} = \frac{AC}{CH - AC} = \frac{AC}{HA},$$

$$\text{and } \frac{A'M}{AC} = \frac{AC + CM}{AC} = \frac{AM + HA}{HA} = \frac{HM}{HA};$$

$$\therefore \frac{HM^2}{HA^2 . A'M^2} = \frac{1}{AC^2},$$

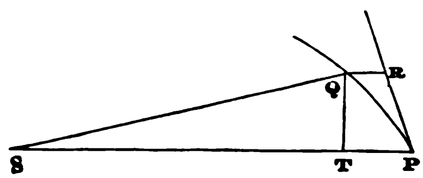
$$\text{and} \therefore F \propto \frac{1}{PM^3}.]$$

## Prop. IX. Prob. IV.

*A body revolves in an equiangular spiral; to find the law of force tending to the centre of the spiral.*

[Def. An equiangular spiral is a curve, in which the tangent at every point makes the same angle with the line joining the point of contact with a certain fixed point called the centre or pole of the spiral: *i. e.* in the figure of this proposition, $SPR$ is a constant angle.]

Let $S$ be the centre of force, $P$ the position of the body at a given time, $Q$ a point in the curve very near to $P$; $PR$




the tangent at $P$, $QR$ parallel to $SP$, and $QT$ perpendicular to $SP$.

$$\text{Then } F \propto \text{limit} \frac{QR}{SP^2 . QT^2}.$$

Now, since the angle $SPR$ is given, if we take $PSQ$ any given small angle, we shall have all the angles in the figure $SPRQT$ given*; and therefore, at whatever point of the orbit we take $P$, the figure $SPRQT$ will be similar, and the lines in

* [Newton appears to assume that if two quadrilaterals have the same angles they will be similar, which is not necessarily the case. It will be easily seen however that the conclusion is correct in this instance, because on account of the law according to which the curve is generated being wholly independent of any linear magnitudes, if $Q'$, $R'$ be the points corresponding to $Q$, $R$ in the adjacent quadrilateral, $QSQ'$ being equal to $PSQ$, the portion of the curve from $Q$ to $Q'$ and the quadrilateral $SQR'Q'$ will differ from the curve from $P$ to $Q$ and the quadrilateral $SPRQ$ only in being drawn upon a different scale, the homologous lines in the two cases being as $SQ : SP$.]

it will be proportional to any one of the homologous lines; hence $QR$ and $QT$ will each be proportional to $SP$, and $\frac{QT^2}{QR}$ to $SP$.

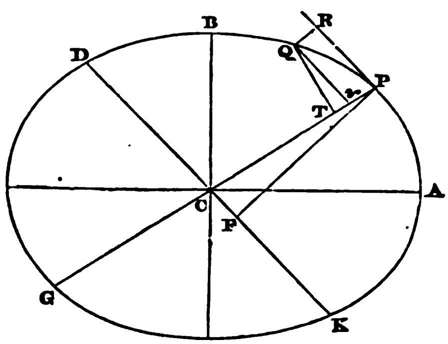
Now let the angle $PSQ$ be indefinitely diminished, then in the limit $QR$ will $\propto PR^2 \propto QT^2$, by Lemma XI. Therefore $\frac{QT^2}{QR}$ will continue in the limit, as before, to be proportional to $SP$;

$$\therefore F \propto \frac{1}{SP^5}.$$

## PROP. X. PROB. V.

*A body describes an ellipse; to find the law of force tending to the centre.*

Let $P$ be the position of the body at any given time, $Q$ a point contiguous to $P$, $PR$ the tangent at $P$, $QT$ perpendi-




cular to the diameter $PCG$, $DCK$ the diameter conjugate to $PCG$, $PF$ perpendicular to $DCK$, $Qv$ an ordinate to $PCG$.

$$\text{Then } F = \frac{2h^2}{CP^2} \frac{QR}{QT^2} \text{ ultimately.}$$

By similar triangles $QTv$, $PFC$,

$$\frac{QT}{Qv} = \frac{PF}{CP}; \therefore CP^2 . QT^2 = Qv^2 . PF^2$$

$$\therefore F = 2h^2 \text{ limit } \frac{QR}{Qv^2 \,.\, PF^2}.$$

$$\text{But } Qv^2 = \frac{CD^2}{CP^2} Pv \,.\, vG \text{ (Conics, Prop. VIII. page 181)}$$

$$= \frac{CD^2}{CP^2} QR \,.\, vG;$$

$$\therefore F = 2h^2 \text{limit } \frac{CP^2}{CD^2 \,.\, PF^2 \,.\, vG}$$

$$= 2h^2 \text{limit} \frac{CP^2}{AC^2 \,.\, BC^2 \,.\, vG} \text{ (Conics, Prop. X. page 183)}$$

$$= \frac{h^2}{AC^2 \,.\, BC^2} \,.\, CP, \text{ since } vG = 2\,CP \text{ ultimately,}$$

$$\text{or } F \propto CP.$$

COR. 1. Therefore the force is proportional to the distance of the body from the centre of the ellipse: and conversely, if the force vary as the distance, the body will move in an ellipse having its centre in the centre of force, or it may be in a circle, which is a particular case of the ellipse.

COR. 2. And the periodic times in all ellipses described round the same centre of force will be the same. For the times will be equal in similar ellipses by Prop. IV. Cors. 3 and 8, and in ellipses having a common major axis they are proportional to the areas of the ellipses directly, and the areas described in a unit of time inversely; that is, as the minor axes directly, and the velocities of the bodies at the extremity of the major axes inversely; that is, as the minor axes directly, and as ordinates drawn to the same point of the common axis inversely; that is, since these ordinates are proportional to the minor axes, in a ratio of equality.

[This result will appear more simply thus:

Suppose $F = \mu CP$, where $\mu$ is a constant quantity depending upon the intensity of the force residing in the centre, and usually called the *absolute force* of the centre: then, by the preceding proposition, $h^2 = \mu AC^2 \,.\, BC^2$.

Let $P$ be the periodic time, that is, the time employed in describing the complete ellipse; then since the area described in a unit of time is $\frac{h}{2}$, and the whole area of the ellipse is $\pi AC \cdot BC$, we shall have

$$P = \frac{2\pi AC \cdot BC}{h} = \frac{2\pi}{\sqrt{\mu}}.$$

Hence the periodic time depends solely upon the intensity of the force in the centre.]

[Cor. 3. Since $P$ is independent of both axes of the ellipse, its value will be the same if we suppose the minor axis to be indefinitely diminished, in which case the motion will approximate to that of a body oscillating in a straight line under the action of an attractive force varying directly as the distance: hence the time of a complete oscillation of a body moving in the manner described will be $\frac{2\pi}{\sqrt{\mu}}$.]

## SCHOLIUM.

If the centre of the ellipse move to an infinite distance, the ellipse will become a parabola and the body will move in this parabola; and the force now tending to a centre infinitely distant will become constant. This is a theorem due to Galileo. And if the parabola be changed into an hyperbola, the body will move in this hyperbola, the force becoming repulsive instead of attractive.

## SECTION III.

### ON THE MOTION OF A BODY IN A CONIC SECTION, ABOUT A CENTRE OF FORCE IN THE FOCUS.

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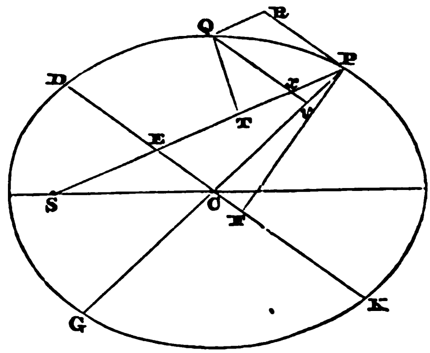
### PROP. XI. PROB. VI.

*A body revolves in an ellipse; to find the law of force tending to one of the foci.*

Let $S$ be the focus of the ellipse, $P$ the position of the body at any given time, $Q$ a contiguous point in the orbit,




$PCG$, $KCD$ conjugate diameters, $PR$ the tangent at $P$, $QR$ parallel to $SP$, $Qxv$ to $PR$, $QT$ perpendicular to $SP$, $PF$ to $CK$, and $E$ the point of intersection of $SP$ and $CD$.

Then, $$F = \frac{2h^2}{SP^2}\frac{QR}{QT^2}, \text{ ultimately.}$$

By similar triangles $QTx$, $PEF$,

$$\frac{QT^2}{Qx^2} = \frac{PF^2}{PE^2} = \frac{PF^2}{AC^2}; \text{ (Conics, Prop. III. Cor. page 175)}$$

and, by similar triangles, $Pxv$, $PEC$,

$$\frac{Px}{Pv} = \frac{PE}{CP} = \frac{AC}{CP},$$

$$\therefore QR = Px = Pv\frac{AC}{CP};$$

and $$\frac{QR}{QT^2} = Pv\frac{AC}{CP}\frac{AC^2}{Qx^2 . PF^2}$$

$$= \frac{Pv}{Qv^2}\frac{AC^3}{CP . PF^2}, \text{ ultimately},$$

$$= \frac{1}{vG}\frac{CP^2}{CD^2}\frac{AC^3}{CP . PF^2} \text{ (Conics, Prop. VIII. page 181)}$$

$$= \frac{AC^3}{2}\frac{1}{CD^2 . PF^2}, \text{ ultimately},$$

$$= \frac{AC^3}{2AC^2 . BC^2} \text{ (Conics, Prop. X. page 183)}$$

$$= \frac{AC}{2BC^2};$$

$$\therefore F = \frac{h^2 AC}{BC^2} \cdot \frac{1}{SP^2} \propto \frac{1}{SP^2}.$$

This result may also be deduced from Prop. VII. Cor. 3. For the force under the action of which the body would describe the ellipse round $C \propto CP$: but

force tending to $C$ : force tending to $S$ :: $CP.SP^2 : PE^3 (AC^3)$,

$$\therefore \text{force tending to } S \propto \frac{1}{SP^2}.$$

[COR. If $\mu$ be the *absolute force* of the centre,

$$\mu = \frac{h^2 . AC}{BC^2} = \frac{2h^2}{L},$$

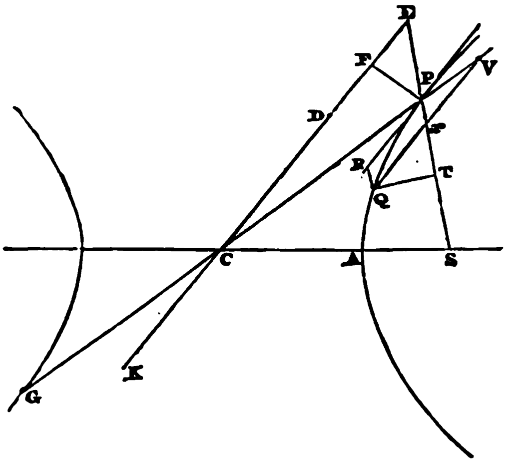
where $L$ is the latus rectum of the ellipse. (Conics, Prop. VI. Cor. 1, page 179.)]

This proposition might be at once extended as in the case of the fifth problem to the hyperbola and parabola; but on account of the importance of the problem it will be worth while to give the demonstrations in full.

### Prop. XII. Prob. VII.

*A body moves in an hyperbola; to find the law of force tending to one of the foci.*

Let $S$ be the focus of the hyperbola, $P$ the position of the body at any given time, $Q$ a contiguous point in the orbit, $PCG$, $DCK$ conjugate diameters, $PR$ the tangent at $P$, $QR$ parallel to $SP$, $Qxv$ to $PR$, $QT$ perpendicular to $SP$, $PF$ to




$CD$ produced, and $E$ the point of intersection of $SP$ and $CD$ produced.

Then, $F = \dfrac{2h^2}{SP^2}\dfrac{QR}{QT^2}$, ultimately.

By similar triangles, $QTx$, $PEF$,

$$\frac{QT^2}{Qx^2} = \frac{PF^2}{PE^2} = \frac{PF^2}{AC^2} \text{ (Conics, Prop. III. Cor. page 190.)}$$

And by similar triangles $Pxv$, $PEC$,

$$\frac{Px}{Pv} = \frac{PE}{CP} = \frac{AC}{CP};$$

$$\therefore QR = Px = Pv\frac{AC}{CP};$$

$$\text{and} \quad \frac{QR}{QT^2} = Pv\frac{AC}{CP}\,\frac{AC^2}{Qx^2 \,.\, PF^2}$$

$$= \frac{Pv}{Qv^2}\,\frac{AC^3}{CP\,.\,PF^2}, \text{ ultimately,}$$

$$= \frac{1}{vG}\,\frac{CP^2}{CD^2}\,\frac{AC^3}{CP\,.\,PF^2} \text{ (Conics, Prop. x. page 198)}$$

$$= \frac{AC^3}{2}\,\frac{1}{CD^2\,.\,PF^2}, \text{ ultimately,}$$

$$= \frac{AC^3}{2AC^2\,.\,BC^2} \text{ (Conics, Prop. xi. Cor. page 198)}$$

$$= \frac{AC}{2BC^2};$$

$$\therefore F = \frac{h^2 AC}{BC^2}\cdot\frac{1}{SP^2} \propto \frac{1}{SP^2}.$$

This result like that of the preceding proposition may be deduced from Prop. VII. Cor. 3.

In the same manner it appears, that if the force be repulsive instead of attractive the body may describe the opposite branch of the hyperbola.

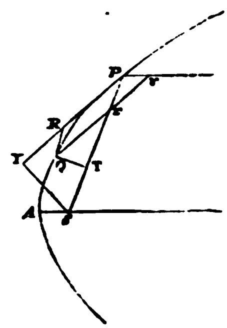
[Cor. As in the case of the ellipse,

$$\mu = \frac{2h^2}{L} \text{ (Conics, Prop. vi. Cor. page 193.)]}$$

## Prop. XIII. Prob. VIII.

*A body moves in a parabola; to find the law of force tending to the focus.*

Let $S$ be the focus of the parabola, $P$ the position of the body at any given time, $Q$ a contiguous point in the orbit, $PRY$ the tangent at $P$, $QR$ parallel to $SP$, $Qxv$ to $PR$, $QT$ perpendicular to $SP$, and $SY$ to $PRY$.




Then $F = \frac{2h^2}{SP^2}\frac{QR}{QT^2}$, ultimately.

But $QR = Px = Pv$ (Conics, Prop. II. page 164)

$= \frac{Qv^2}{4SP}$ (Conics, Prop. IX. page 169)

$= \frac{Qx^2}{4SP}$, ultimately;

$\therefore \frac{QR}{QT^2} = \frac{1}{4SP}\frac{Qx^2}{QT^2} = \frac{1}{4SP}\frac{SP^2}{SY^2}$, by similar triangles,

$$= \frac{SP}{4AS \cdot SP} \text{ (Conics, Prop. VI. Cor. page 167)}$$

$$= \frac{1}{4AS};$$

$$\therefore F = \frac{h^2}{2AS} \cdot \frac{1}{SP^2} \propto \frac{1}{SP^2}.$$

COR. 1. It follows from the last three propositions, that if a body $P$ be projected from any point $P$ with a given velocity in a given direction $PR$, and be acted upon by a centripetal force varying inversely as the square of the distance from the centre, the body will move in a conic section having the centre of force in its focus; and conversely. For the focus, the point of contact, and the position of the tangent being given, a conic section can be described, which shall have a given curvature at that point. But the curvature is given because the force and the velocity of the body are given; and two orbits touching each other cannot be described with the same centripetal force and the same velocity.

COR. 2. If the velocity, with which the body leaves $P$, be that with which the small line $PR$ can be described in a certain very short space of time; and if the force be able during the same time to move the body through the space

## PROP. XV. THEOR. VII.

*On the same hypothesis, the squares of the periodic times in ellipses are proportional to the cubes of the major axes.*

Let $P$ be the periodic time in one of the ellipses, then

$$P = \frac{2 \text{ area of the ellipse}}{h}$$

$$= \frac{2\pi AC \cdot BC}{h},$$

$$\therefore P^2 = \frac{4\pi^2 AC^2 \cdot BC^2}{\mu BC^2} AC \text{ (Prop. xi. Cor.)}$$

$$= \frac{4\pi^2}{\mu} AC^3 \propto AC^3.$$

COR. Hence the periodic time in an ellipse is the same as in a circle, the diameter of which is the major axis of the ellipse.

## PROP. XVI. THEOR. VIII.

*On the same hypothesis, the velocity in any of the orbits varies inversely as the perpendicular from the focus on the tangent and directly as the square root of the latus rectum.*

For if $v$ be the velocity, and $p$ the perpendicular from the focus on the tangent, we have seen (Prop. I. Cor. 1) that

$$vp = h,$$

$$\text{but } h \propto \sqrt{L};$$

$$\therefore v \propto \frac{\sqrt{L}}{p}. \quad \text{Q.E.D.}$$

COR. 1. The latera recta of the orbits are in the ratio compounded of the duplicate ratio of the perpendiculars and the duplicate ratio of the velocities.

[In other words $L \propto v^2p^2$.]

$QR$; then the body will move in a conic section, of which the *latus rectum* will be the ultimate value of the ratio $\frac{QT^2}{QR}$. In these corollaries the circle may be included as a particular case of the ellipse, and the case in which the body falls directly to the centre is excluded.

[COR. 3. As in the preceding cases,

$$\mu = \frac{2h^2}{L} \text{ (Conics, Prop. I. page 163.)}]$$

## PROP. XIV. THEOR. VI.

*If any number of bodies revolve about a common centre, and the force vary inversely as the square of the distance, the latera recta of the orbits described will be as the squares of the areas described in equal times.*

For we have seen in each of the three preceding propositions, that

$$\mu = \frac{2h^2}{L},$$

where $\mu$ depends upon the absolute intensity of the central force; if therefore this be given,

$$L \propto h^2,$$

and since $h$ is twice the area described in a unit of time, which may be any given time, the proposition is true. Hence if any number of bodies, &c. Q.E.D.

COR. Hence the whole area of the ellipse, or the rectangle under the axes which is proportional to it (page 339, note), varies in a ratio compounded of the subduplicate ratio of the *latus rectum*, and the ratio of the periodic time.

That is, if $P$ be the periodic time, the area $\propto L^{\frac{1}{2}}P$; for $h \propto L^{\frac{1}{2}}$, and since the areas described are proportional to the time, the area of the ellipse $= \frac{h}{2}P$, and $\therefore \propto L^{\frac{1}{2}}P$.

the focus; in the ellipse it varies more than in this ratio; and in the hyperbola less. For, in the parabola,

$$SY^2 \propto SP \text{ (Conics, Prop. VI. Cor. page 167.)}$$

In the ellipse,

$$SY^2 \propto \frac{SP}{2AC - SP} \text{ (Conics, Prop. IV. Cor. page 176);}$$

and in the hyperbola,

$$SY^2 \propto \frac{SP}{2AC + SP} \text{ (Conics, Prop. IV. Cor. page 191.)}$$

Now in this expression, as the numerator increases the denominator also increases, therefore $SY^2$ varies less than $SP$; and in like manner in the ellipse it varies more.

[This result may be conveniently expressed thus:

$$\text{in the parabola, } v^2 = \frac{2\mu}{SP},$$

$$\text{in the ellipse, } v^2 = \frac{\mu}{SP}\left(2 - \frac{SP}{AC}\right),$$

$$\text{in the hyperbola, } v^2 = \frac{\mu}{SP}\left(2 + \frac{SP}{AC}\right).]$$

COR. 7. In the parabola, the velocity of the body at any distance from the focus, is to the velocity of a body revolving in a circle at the same distance as the square root of 2 to 1; in the ellipse it is less, and in the hyperbola it is greater than in this ratio. For, by Cor. 2, the velocity at the vertex of a parabola is in this proportion, and by Cor. 6, and Prop. IV. Cor. 6, the same proportion is preserved for all distances. Hence also in the parabola, the velocity is every where equal to that in a circle at half the distance, in the ellipse less, and in the hyperbola greater.

[This will perhaps appear more clearly thus. In the expression for the velocity in the ellipse put $AC = SP$, the velocity then becomes that in a circle, which call $V$ as before;

$$\therefore\ V^2 = \frac{\mu}{SP};$$

$$\begin{aligned} \therefore \text{ in the parabola } \quad & v = V\sqrt{2}, \\ \text{in the ellipse } \quad & v < V\sqrt{2}, \\ \text{in the hyperbola } \quad & v > V\sqrt{2}. \end{aligned}$$

Also if $R$ be the radius of a circle in which the velocity would be the same as at the point $P$ of the parabola, we must have

$$\frac{2\mu}{SP} = \frac{\mu}{R}, \text{ or } R = \frac{SP}{2}.]$$

COR. 8. The velocity of a body revolving in any conic section is to the velocity in a circle at the distance of half the *latus rectum*, as that distance is to the perpendicular from the focus on the tangent.

[For $v^2 = \dfrac{\mu L}{2p^2}$, and if $V$ be the velocity in the circle

$$V^2 = \frac{2\mu}{L};$$

$$\therefore\ v^2 : V^2 :: \frac{L^2}{4} : p^2,$$

$$\text{or } v : V :: \frac{L}{2} : p.]$$

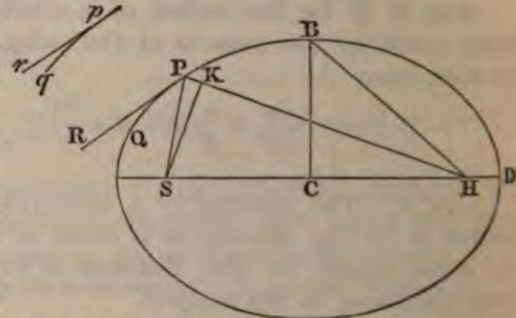
COR. 9. And hence, since (by Prop. IV. Cor. 6) the velocities of bodies revolving in circles are in the inverse subduplicate ratio of their distances, the velocity of a body in a conic section will be to the velocity in a circle at the same distance, as a mean proportional between that common distance and the semi-latus rectum to the perpendicular on the tangent.

$$[\text{For} \quad v^2 = \frac{\mu L}{2p^2} = \frac{\mu L}{2SY^2}, \quad V^2 = \frac{\mu}{SP};$$

$$\therefore\ v^2 : V^2 :: \frac{L}{2} . SP : SY^2.]$$

## Prop. XVII. Prob. IX.

*Given that the centripetal force is inversely proportional to the square of the distance from the centre, and that the absolute force of the centre is known; it is required to find the curve, which will be described by a body which is projected from a given point with a given velocity in a given direction.*




Let the force tending to $S$ be that under the action of which a body $p$ would describe any given orbit $pq$, and let the velocity of this body at the point $p$ be known. Let the body $P$ start from the point $P$ with the given velocity in the direction $PR$, and let it be made by the action of the centripetal force to move in the conic section $PQ$. Therefore $PR$ will be a tangent to the curve at $P$. Let $pr$ in like manner touch the orbit $pq$ in $p$, and if from $S$ perpendiculars be supposed to be drawn to these tangents, the latus rectum of the conic section $(PQ)$ will be to the latus rectum of the orbit $pq$ in the ratio compounded of the duplicate ratio of the perpendiculars and the duplicate ratio of the velocities, (Prop. XVI. Cor. 1), and therefore is given. Let $L$ be the latus rectum of the conic section.

Moreover the focus $S$ of the conic section is given. Take $RPH$ equal to the supplement of $RPS$; and the other focus $H$ manifestly lies on the line $PH$ thus determined in position. On $PH$ let fall the perpendicular $SK$, and suppose the semi-axis minor $BC$ to be drawn, then we shall have

$$SP^2 - 2KP \cdot PH + PH^2 = SH^2 = 4CH^2$$
$$= 4BH^2 - 4BC^2$$
$$= (SP + PH)^2 - L(SP + PH)$$
$$\text{(since } 2BH = 2AC = SP + PH, \text{ and } L \cdot AC = 2BC^2)$$
$$= SP^2 + 2SP \cdot PH + PH^2 - L(SP + PH);$$

$$\therefore L(SP + PH) = 2SP \cdot PH + 2KP \cdot PH$$
$$= (2SP + 2KP)\, PH$$
$$\text{or } SP + PH : PH :: 2SP + 2KP : L.$$

Here $L$, $SP$, and $KP$ are known, therefore $PH$ is known.

Also, if the velocity of the body at $P$ be such that $L$ is less than $2SP + 2KP$, $PH$ will lie on the same side of the tangent $PR$ as the line $SP$; and therefore the figure will be an ellipse, and the foci $S$, $H$ having been found, and also the major axis $SP + PH$, the ellipse will be entirely determined.

If the velocity be such that $L = 2SP + 2PK$, $PH$ will be infinite; and the figure will be a parabola having its axis $SH$ parallel to $PK$, and therefore will be determined.

Lastly, if the velocity be still greater than in the preceding case, $PH$ must be taken on the opposite side of the tangent; and the tangent thus passing between the foci, the figure will be an hyperbola having its major axis equal to $SP \sim PH$, and therefore will be determined. For if the body in these several cases were to revolve in the conic section so found, it has been shewn in Props. XI. XII. and XIII. that the centripetal force would be inversely proportional to the square of the distance from $S$; and therefore the curve $PQ$ is rightly determined, which the body under the action of such a force would describe if projected from the given point $P$, with a given velocity, and in a given direction.

COR. 1. Hence in every conic section, if the vertex $D$, the latus rectum $L$, and the focus $S$ be given, the other focus $H$ is given by taking

$$DH : DS :: L : 4DS - L.$$

For the proportion

$$SP + PH : PH :: 2SP + 2KP : L$$

becomes, in the case of this corollary,

$$DS + DH : DH :: 4DS : L,$$
$$\text{and } \therefore DS : DH :: 4DS - L : L.$$

Cor. 2. Wherefore if the velocity at the vertex $D$ be given, the orbit may be immediately found by taking the latus rectum $L$, such that

$L : 2DS ::$ square of given velocity : square of velocity in circle of radius $DS$, (Prop. XVI. Cor. 3)
and then taking

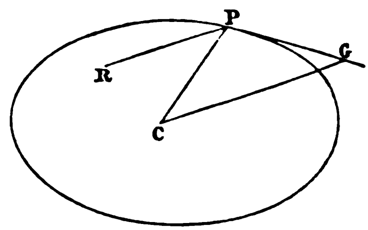
$$DH : DS :: L : 4DS - L.$$

Cor. 3. Hence also if the body move in any conic section, and be disturbed from its orbit by any extraneous impulse, the subsequent orbit may be determined. For by compounding the motion of the body with that motion which the impulse above would generate, we shall know the magnitude and direction of the motion with which the body will leave the point at which the impulse takes place.

Cor. 4. And if the body be disturbed by any continuous extraneous force, we can determine its course approximately by calculating the changes which the force produces at certain points, and concluding from analogy the change which takes place in the intermediate parts of the orbit.

SCHOLIUM.

If a body $P$ move in any conic section, whose centre is $C$, under the influence of a force tending to any point $R$, and it be required to find the law of force; draw $CG$ parallel to $RP$ and meeting the tangent $PG$ in $G$; then by Prop. VII. Cor. 3,




force tending to $R$ : force tending to $C$ :: $CG^3 : CP \,.\, RP^2$,

$$\text{or force tending to } R \propto \frac{CG^3}{RP^2}.$$

[The proof of the proposition, that a body under the action of a central force varying in intensity as the inverse

square of the distance will move in a conic section, may be given in the following direct and elegant manner; for the knowledge of which I am indebted to my friend R. L. Ellis, Esq. of Trinity College.

Lemma. In any central orbit the angular velocity of the body varies inversely as the square of the distance of the body from the centre.

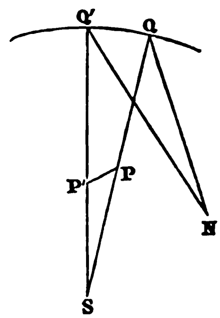
This follows at once from Newton, Prop. I., in the same manner as Cor. 1 follows from that proposition. For by the angular velocity we mean the velocity with which the line joining the body with the centre revolves, and therefore as in Fig. Prop. I., the velocity at $A \propto AB$, so the angular velocity will $\propto$ angle $ASB$.

$$\text{But area } ASB = \frac{AS \,.\, SB}{2} \sin ASB$$

$$= \frac{AS^2}{2} \,.\, ASB \text{ ultimately,}$$

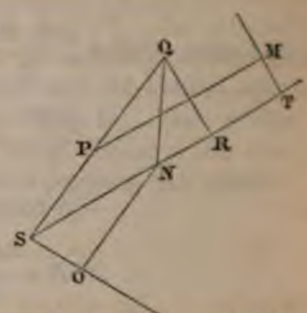
$$= \text{a constant quantity;}$$

$$\therefore\ ASB \propto \frac{1}{AS^2}. \qquad \text{Q.E.D.}$$




This being premised, let $S$ be a centre of force, $PP'$ a very small portion of an orbit described about $S$ in a given indefinitely small time; draw the lines $SPQ$, $SP'Q'$, and with centre $S$ and any given radius describe a circle cutting these two lines in $Q$ and $Q'$ respectively. Then by the preceding Lemma the angle $QSQ'$, or the arc $QQ'$ (since the radius $SQ$ is given) varies inversely as the square of $SP$, and therefore directly as the force upon $P$ tending towards $S$. Consequently $QQ'$ may be taken to represent the velocity generated by the central force while the body moves from $P$ to $P'$; the direction of $QQ'$, it will be observed, is perpendicular to the direction

of the force. Now take the line $QN$, in the direction perpendicular to that of the body's motion, that is, perpendicular to $PP'$, and representing the velocity of the body when at $P$ on the same scale that $QQ'$ represents the velocity generated by the force while the body passes from $P$ to $P'$; join $Q'N$; then since $QN$ represents the velocity at $P$ in magnitude, but is in direction perpendicular to that velocity, and $QQ'$ represents the velocity generated in passing from $P$ to $P'$ in magnitude, but is in direction perpendicular to the generating force, therefore compounding these velocities $Q'N$ will represent the velocity at $P'$, but will be in the direction perpendicular to that velocity. Hence $N$ will be a fixed point, and lines $QN$, $Q'N$, &c. drawn as above will represent the velocity throughout the motion.




Now through $S$ draw $SO$ perpendicular to $SP$, through $Q$ draw $QR$ perpendicular to $SN$ produced if necessary, and through $N$ draw $NO$ perpendicular to $SO$. Then it is easy to see that $QR$ will represent in magnitude the resolved part of the body's velocity parallel to $SN$, since the velocity $NQ$ may be supposed to be resolved into the two $QR$ and $NR$; also $SO$ will represent in magnitude the velocity parallel to $SP$. But the triangles $SQR$, $NSO$ are similar, and therefore $QR : SO :: SQ : SN$, which is a constant ratio; or the velocity of the body parallel to $SN$ bears a constant ratio to its velocity parallel to $SP$.

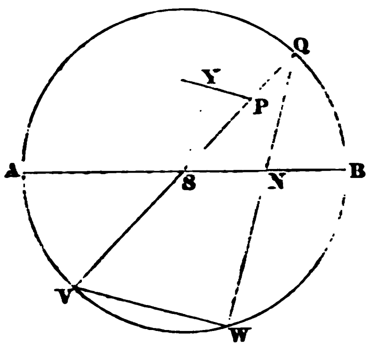
If then in $SN$, produced if necessary, we take a point $T$, such that the distance from $P$ of a line drawn through $T$ perpendicular to $ST$ is in this ratio to $SP$, that is, in the figure, if $T$ be such that

$$PM : SP :: SQ : SN,$$

it follows that throughout the motion the distances of the particle from $S$ and from this line $TM$ are in a constant ratio to each other; that is, the path of the body is a conic section of which $S$ is the focus and $TM$ the directrix. Q.E.D.

The latter part of this proposition may also be exhibited in the following form, which, though less elegant than that which precedes, may nevertheless be worthy of the student's attention.

Suppose it to be proved, as before, that $QN$ represents the velocity. Then let the circle of which the centre is $S$ and radius $SQ$ be described, and let $QS$, $QN$ meet it in $V$ and $W$ respectively; also through $P$ draw $PY$ perpendicular to $QN$, and $SY$ perpendicular to $PY$. Then it has been proved that $QN$ represents the velocity, also $PY$ being perpendicular to $QN$ is a tangent to the body's path,




$$\therefore\ QN \,.\, SY = h. \quad \text{(Newton, Prop. I. Cor. I.)}$$

Also if $QN$ produced meet the circle in $A$ and $B$, we have

$$QN \,.\, NW = NB \,.\, NA = SB^2 - SN^2 = SQ^2 - SN^2.$$

Also by similar triangles $QWV$, $SYP$,

$$\frac{SY}{SP} = \frac{QN + NW}{2SQ} = \frac{QN^2 + QN \,.\, NW}{2SQ \,.\, QN} = \frac{QN^2 + SQ^2 - SN^2}{2SQ \,.\, QN}.$$

$$\therefore\ \frac{2h \,.\, SQ}{SP} = \frac{2QN \,.\, SY \,.\, SQ}{SP} = QN^2 + SQ^2 - SN^2$$

$$= \frac{h^2}{SY^2} + SQ^2 - SN^2.$$

Now there are three cases to consider.

(1) Suppose $SN = SQ$;

$$\therefore\ SY^2 = \frac{h}{2SQ} \,.\, SP.$$

Comparing this with the result of Prop. VI. Cor. page 167,

we see that the curve is a parabola of which the latus rectum is $\frac{2h}{SQ}$.

(2) Suppose $SN < SQ$;

$$\therefore \frac{SP}{SY^2} = \frac{2SQ}{h} - (SQ^2 - SN^2)\frac{SP}{h^2},$$

$$\frac{SY^2}{SP} = \frac{h^2}{2h \cdot SQ - (SQ^2 - SN^2)\, SP} = \frac{\dfrac{h^2}{SQ^2 - SN^2}}{\dfrac{2h \cdot SQ}{SQ^2 - SN^2} - SP}.$$

Comparing this with the result of Prop. IV. Cor. p. 176, we see that the curve is an ellipse, in which

$$BC^2 = \frac{h^2}{SQ^2 - SN^2}, \text{ and } AC = \frac{h \cdot SQ}{SQ^2 - SN^2}.$$

(3) Suppose $SN > SQ$;
then, in like manner,

$$\frac{SY^2}{SP} = \frac{\dfrac{h^2}{SN^2 - SQ^2}}{\dfrac{2h \cdot SQ}{SN^2 - SQ^2} + SP},$$

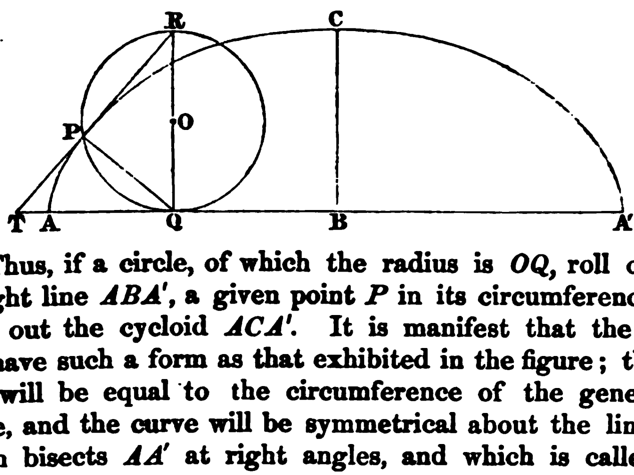
comparing which with Prop. IV. Cor. p. 191, we see that the curve is an hyperbola.]

# APPENDIX

## CONTAINING THE THEORY OF CYCLOIDAL OSCILLATIONS.

1. It is the purpose of the following articles to solve the problem of finding the time of oscillation of a heavy particle, when constrained to move upon the arc of a cycloid. But before giving the solution, it will be necessary to define the cycloid, and to investigate some of its properties.

2. Def. *A cycloid is the curve traced out by a point in the circumference of a circle, which rolls upon a given straight line.*



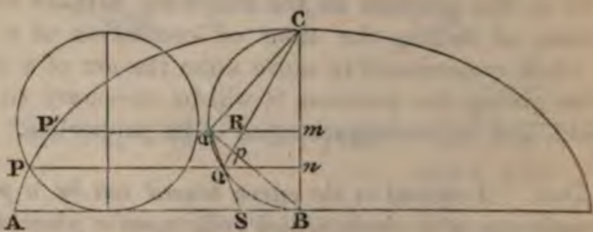

Thus, if a circle, of which the radius is $OQ$, roll on the straight line $ABA'$, a given point $P$ in its circumference will trace out the cycloid $ACA'$. It is manifest that the curve will have such a form as that exhibited in the figure; the line $AA'$ will be equal to the circumference of the generating circle, and the curve will be symmetrical about the line $BC$, which bisects $AA'$ at right angles, and which is called the axis of the cycloid.

3. *To draw a tangent to a cycloid.*

Join $PQ$, $Q$ being the point of the generating circle in contact with $AA'$ at any given moment; then the generating circle moves into its next position by turning about $Q$, and therefore the motion of $P$ will be for a very short space of time the same as if it were describing a circle about $Q$, that is, its motion will be perpendicular to $PQ$.

Hence the tangent at $P$ will be perpendicular to $PQ$, and will therefore pass through $R$ the other extremity of the diameter $QOR$.

4. *To find the length of the arc of a cycloid.*




Let $P$, $P'$ be two contiguous points in a cycloid; on the axis $BC$ describe a semicircle, and through $P$, $P'$ draw the lines $Pn$, $P'm$ perpendicular to $BC$ and cutting the semicircle in $Q$ and $Q'$ respectively. Join $CQ$, $CQ'$, $BQ'$, $Q'Q$, and produce the last to meet $AB$ in $S$; also let $R$, $p$ be the intersections of $CQ$, $P'm$, and $CQ$, $Q'B$ respectively.

Then, when $P'$ approaches indefinitely near to $P$, $QR$ being parallel to the tangent at $P$ will be ultimately parallel, and therefore also equal, to $PP'$. Also $Q'B$ will be ultimately perpendicular to $CQ$, and therefore $Qp$ will be ultimately $CQ - CQ'$.

$$\text{Now } SQ' \text{ ultimately} = SB;$$

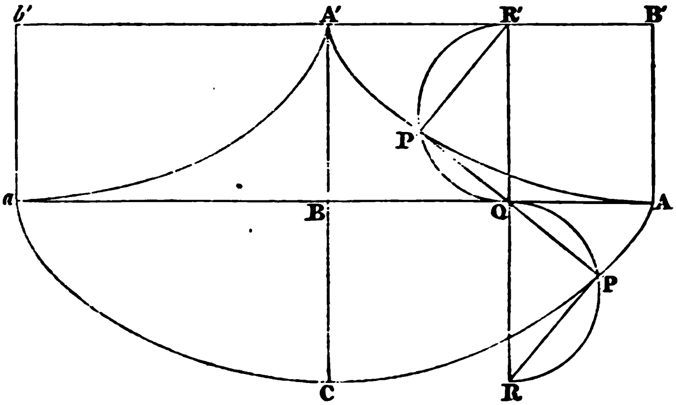
$$\therefore \text{angle } SQ'B = SBQ' = mQ'B;$$

$\therefore$ in the triangles $QQ'p$, $RQ'p$, we have $\angle QQ'p = \angle RQ'p$, and $\angle Q'pQ = \angle Q'pR$, (each being a right angle,) and the side $Q'p$ common; $\therefore Qp = Rp$, and $\therefore QR = 2Qp$, or $PP'$ ultimately $= 2(CQ - CQ'.)$

But $PP'$ is the increment of the arc of the cycloid in passing from the point $P'$ to the contiguous point $P$, and $CQ - CQ'$ is the corresponding increment of the chord $CQ'$, which is equal to the chord of the generating circle touching the cycloid at $P$; hence *the arc of the cycloid measured from*

*the vertex to any point equals twice the chord of the generating circle which touches the curve at that point.* In the figure of Art. 2, $CP = 2PR$,

5. *To make a pendulum oscillate in a given cycloid.*




Let $APC$ be a given semicycloid, having base $AB$ and axis $BC$; produce $CB$ to $A'$, making $BA' = BC$, and complete the rectangle $A'BAB'$: with $A'B'$ as base, and $AB'$ as axis, describe the semicycloid $AP'A'$.

Take any line $R'QR$ equal and parallel to $A'BC$, and on $RQ$, $R'Q$ describe the two generating semicircles $QPR$, $QP'R'$; join $QP$, $PR$, $QP'$, $P'R'$.

Then the circular arc $QP = AQ$, as is manifest from the mode in which the cycloid is generated; and in like manner, arc $QPR = AB$;

$$\therefore \text{arc } PR = BQ = A'R' = \text{arc } P'R';$$

$$\therefore PR = P'R',$$

$$\text{also, } QR = QR',$$

and angle $QPR$ = angle $QP'R'$, each being a right angle;

$\therefore$ the triangles $QPR$, $QP'R'$ are equal in all respects.

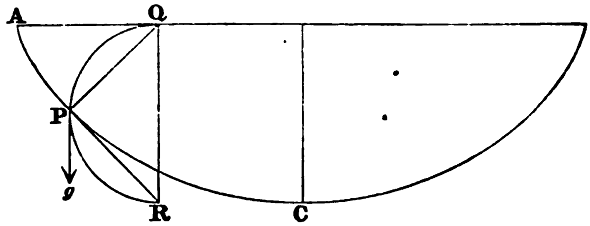
Hence angle $PQR = P'QR'$; $\therefore PQP'$ is a straight line.

Also $PP'$ (which is a tangent to $A'P'A$ at $P'$,)

$$= 2P'Q = \text{arc } P'A.$$

Hence if a string of length $A'P'A$, fixed at $A'$, and wrapped upon the semicycloid $A'P'A$, be unwrapped, beginning at $A$, a particle attached to its extremity will trace out the semicycloid $APC$. And by means of another semicycloid $A'a$, the particle may be made to describe the other half of the cycloid $ACa$.

6. *To find the time of oscillation of a heavy particle moving on the surface of a cycloid.*




Let $P$ be the position of the particle at any time, $QPR$ the corresponding position of the generating circle, $PR$ the tangent at $P$, $C$ the lowest point of the cycloid. Then the force, which accelerates or retards the motion of the particle, is the resolved part of the force of gravity in the direction of the tangent, that is, in the direction of $PR$. But gravity acts parallel to $QR$, therefore the resolved part of gravity in the direction of $PR$

$$= g\cos PRQ = g\frac{PR}{QR} = \frac{g}{2}\frac{PC}{QR}, \text{ (by the property of the cycloid)},$$

$$= \frac{g}{4a}.PC,$$

if we call the radius of the generating circle $a$.

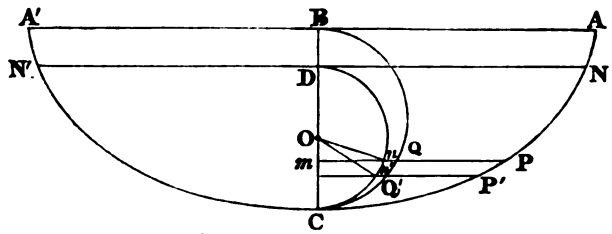
Hence the particle is always acted upon by a force tending to draw it towards $C$ and proportional to $PC$, and will therefore oscillate in the same manner as a particle under the action of a central force varying directly as the distance: therefore by Newton, (Prop. x. Cor. 3),

$$\text{time of oscillation} = 2\pi\sqrt{\frac{4a}{g}} = 4\pi\sqrt{\frac{a}{g}}.$$

6 (bis). This result may be obtained independently of the proposition cited from Newton as follows:

Let $ACA'$ be the cycloid having its axis $BC$ vertical, $N$ a point from which a heavy particle is allowed to descend; then if we draw $NDN'$ horizontal, $N'$ will be the point to which the particle will ascend. On $BC$ describe the semicircle $BQC$, and on $DC$ the semicircle $DnO$ having $O$ for its centre.

Let $P$ be the place of the particle at any given time, $P'$ its place an indefinitely short time after it has passed $P$;




through $P$, $P'$ draw the horizontal lines $PQnm$, $P'Q'n'$, cutting the circles above described in $Q$, $Q'$ and $n$, $n'$ respectively. And join $On$, $On'$.

Denote the angle $COn'$ by $\alpha$, and $nOn'$ by $\theta$, the radius $OC$ by $r$, and the radius of the generating circle by $a$.

Then the velocity of the particle at $P = \sqrt{2g \, . \, Dm}$ (Art. 45, page 312.)

$$\therefore \text{ time of describing } PP' = \frac{PP'}{\sqrt{2g \, . \, Dm}}.$$

But $PP' = 2(CQ - CQ')$, Art. 4,

and if we suppose $BQ$, $CQ$ to be joined, we have from similar triangles $BCQ$, $QCm$,

$$BC : CQ :: CQ : Cm;$$

$$\therefore CQ^2 = BC \, . \, Cm,$$

in like manner $\quad Cn^2 = DC \, . \, Cm;$

$$\therefore \frac{CQ^2}{Cn^2} = \frac{BC}{DC} = \frac{a}{r},$$

and $CQ = \sqrt{\frac{a}{r}}\, Cn = \sqrt{ar}\,\text{chd}\,(\alpha + \theta) = 2\sqrt{ar}\sin\frac{\alpha+\theta}{2}$,

$$\text{similarly } CQ' = 2\sqrt{ar}\sin\frac{\alpha}{2},$$

$$\text{also } Dm = r + r\cos(\alpha+\theta) = 2r\cos^2\frac{\alpha+\theta}{2};$$

$$\therefore \text{ time of describing } PP' = \frac{4\sqrt{ar}\left(\sin\frac{\alpha+\theta}{2} - \sin\frac{\alpha}{2}\right)}{\sqrt{4gr}\cos\frac{\alpha+\theta}{2}},$$

$$= 4\sqrt{\frac{a}{g}}\cdot\frac{\cos\left(\frac{\alpha}{2}+\frac{\theta}{4}\right)\sin\frac{\theta}{4}}{\cos\left(\frac{\alpha}{2}+\frac{\theta}{2}\right)},$$

$$= \sqrt{\frac{a}{g}} \times \theta, \text{ since } \theta \text{ is indefi-}$$

nitely small.

It will be seen that all the small angles such as $\theta$, corresponding to the small arcs $PP'$ between $N$ and $N'$, will together make up four right angles, hence the time of describing the arc $NCN'$ will be $2\pi\sqrt{\frac{a}{g}}$, and the time of describing the arc $NCN'$ and returning to the point $N$ will be $4\pi\sqrt{\frac{a}{g}}$, as before.

Cor. If we have a particle suspended by a string of length $l$, and made to oscillate in a cycloid by the artifice explained in a preceding proposition, then $l = 4a$, and the time of oscillation

$$= 2\pi\sqrt{\frac{l}{g}}.$$

It is to be observed, that by the time of oscillation is meant

the time which elapses between the departure of the particle from the highest point and its return to the same.

7. When the oscillations of a pendulum are very small, we may consider the time of oscillation to be the same as if the extremity described a cycloidal arc; hence if $l$ be the length of a pendulum we may say in general, that, provided the oscillation be small, the time*

$$= 2\pi\sqrt{\frac{l}{g}}.$$

The time of oscillation of a pendulum is an element which can be observed with very great accuracy; hence the observation of a pendulum affords the best means of determining the value of the quantity $g$. Suppose we find by experiment the length of a pendulum which will make a semioscillation in 1″, and let $L$ be its length, then we have

$$\pi\sqrt{\frac{L}{g}} = 1;$$

$$\therefore g = L\pi^2.$$

Such a pendulum is called a seconds pendulum. By this means it is ascertained that the accelerating force of gravity, though nearly the same over the earth's surface, is not accurately so; in fact, observations of the variation of gravity by means of the pendulum may be employed for the purpose of determining the form of the earth. The length of the seconds pendulum, speaking without extreme accuracy, may be said to vary from the poles to the equator between the limits $39\frac{1}{5}$ and 39 inches. In the latitude of London the length is about $39\frac{1}{8}$ inches.

8. A pendulum consisting of a particle suspended by an indefinitely fine string, such as that which we have been considering, is called a *simple* pendulum. But in practice, no pendulum can be made so nearly to fulfil these conditions as to be regarded as a simple pendulum; to deduce the length of

* The time of a very small oscillation on any curve may be found, by taking for $l$ the value of the radius of curvature of the curve at the point on which the oscillations take place.

the theoretical simple pendulum from a seconds pendulum of complicated construction, requires much ingenuity, as well as the application of more complicated mathematical processes than any introduced into this work. It must suffice here to state, that the problem admits of solution to the utmost degree of accuracy.

9. *To find the number of seconds which a pendulum will lose in a day, when lengthened by a given small quantity, supposing the pendulum to be previously a seconds pendulum.*

Let $a$ be the additional length, and $T$ the time of a semi-oscillation, $x$ the number of seconds lost in 24 hours.

$$\text{Then } T = \pi\sqrt{\frac{L+a}{g}} = \pi\sqrt{\frac{L}{g}}\left(1+\frac{a}{2L}\right), \text{ nearly,}$$

$$= 1 + \frac{a}{2L}, \text{ since } \pi\sqrt{\frac{L}{g}} = 1 \text{ by hypothesis.}$$

$$\therefore x = 24 \times 60 \times 60 - \frac{24 \times 60 \times 60}{T},$$

$$= 24 \times 60 \times 60 \times \frac{a}{2L}, \text{ nearly.}$$

Suppose for instance that $\frac{a}{L} = \frac{1}{100}$, then the number of seconds lost $= \frac{24 \times 60 \times 60}{200} = 432$.

10. Observation of the number of beats lost by a seconds pendulum at the summit of a mountain enables us to determine the height of the mountain. For let $x$ be the height of the mountain in feet, $n$ the number of beats lost in 24 hours by a pendulum which vibrates seconds on the earth's surface, $g'$ the value of the accelerating force of the earth's attraction at the summit, then, taking the earth's radius as 4000 miles,

$$g' = g\left(\frac{4000 \times 1760 \times 3}{4000 \times 1760 \times 3 + x}\right)^2,$$

(since the force of gravity varies inversely as the square of the distance from the earth's centre;)

$\therefore$ the time of oscillation at the summit

$$= \pi \sqrt{\frac{L}{g'}} = \pi \sqrt{\frac{L}{g}} \left(1 + \frac{x}{4000 \times 1760 \times 3}\right),$$

$$= 1 + \frac{x}{4000 \times 1760 \times 3}, \text{ since } \pi \sqrt{\frac{L}{g}} = 1.$$

But since the pendulum loses $n$ beats in 24 hours, the time of oscillation

$$= \frac{24 \times 60 \times 60}{24 \times 60 \times 60 - n} = 1 + \frac{n}{24 \times 60 \times 60} \text{ nearly;}$$

$$\therefore x = n \frac{4000 \times 1760 \times 3}{24 \times 60 \times 60},$$

$$= n \times 245 \text{ nearly.}$$

Suppose for example $n = 10$, then the height of the mountain would be 2450 feet.

The same method is applicable to the determination of the depth of a mine; but in this case we must consider that the force of gravity is directly proportional to the distance from the centre of the earth, a proposition not here proved.

11. The demonstration of Art. 6 (bis) may be put in a neater form, as follows*.

In the figure let the point in which the ordinate $P'Q'n'$ meets $BC$ be called $m'$, and suppose $m'n$ to be joined.

Then as before, time of describing $PP' = \dfrac{PP'}{\sqrt{2g \cdot Dm}}$.

$$\text{But} \quad PP' = 2(CQ - CQ'),$$

$$\text{and} \quad CQ^2 = BC \cdot Cm,$$

$$CQ'^2 = BC \cdot Cm',$$

* This addition was made after the Article to which it refers had been printed; otherwise it would have immediately followed that Article.

$$\therefore (CQ - CQ')(CQ + CQ') = BC \, . \, mm',$$

$$\therefore \text{ ultimately } PP' = \frac{BC}{CQ} \, . \, mm' = mm' \sqrt{\frac{BC}{Cm}} \, .$$

And therefore time of describing $PP'$

$$= \sqrt{\frac{BC}{2g}} \, . \, \frac{mm'}{\sqrt{Cm \, . \, Dm}} = \sqrt{\frac{a}{g}} \, . \, \frac{mm'}{mn} \, .$$

Now consider the quadrilateral $Onn'm'$; the angle at $m'$ is a right angle, and the angle at $n$ is ultimately a right angle, and therefore $Onn'm'$ is ultimately a quadrilateral inscribable in a circle; hence

$$nOn' = nm'n' = mnm' = \frac{mm'}{mn} \text{ ultimately.}$$

$$\therefore \text{ time of describing } PP' = nOn' \sqrt{\frac{a}{g}},$$

$$\text{and the time of an oscillation} = 4\pi \sqrt{\frac{a}{g}} \text{ as before.}$$

# HYDROSTATICS.

# HYDROSTATICS.

1. A FLUID is a collection of material particles, which can be moved among each other by an indefinitely small force.

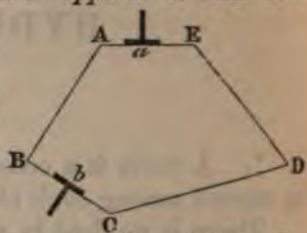
There is no fluid in nature which strictly fulfils the definition we have given; nevertheless those substances, which we shall consider as fluid, fulfil it sufficiently nearly to make the conclusions founded on the definition practically correct.

2. Fluids are distinguished into *elastic* and *non-elastic*. The former class consists of those, the volume of which can be diminished by pressure, and which have an internal expansive force, in virtue of which their volume increases when not constrained by external pressure. Of this kind is *air*, and generally all *gaseous* fluids. The latter class consists of those which have not this property, and the volume of which remains the same whatever pressure they may be subjected to. Of this kind is *water*, and generally all those fluids which we term *liquid*.

These two classes of fluids are also spoken of as *compressible* and *incompressible*. Strictly speaking no known fluid is incompressible, but all ordinary liquids are sufficiently nearly so to enable us to regard them as such without sensible error. Water and even mercury have been compressed by actual experiment, and the amount by which their volume is decreased for a given pressure has been ascertained.

3. As the science of Force, considered as acting on a material particle, or a system of material particles rigidly connected, divided itself into the two sciences of Statics and Dynamics, so in considering the action of force on a fluid the science will be that of *Hydrostatics* or *Hydrodynamics*, according as motion is or is not produced. The mathematical difficulty of the latter science, however, will confine us strictly to the case of a fluid at rest.

4. The characteristic property of fluids is that of *transmitting equally in all directions pressures applied at their surfaces.* Thus, suppose the figure to represent a horizontal section of a vessel containing fluid, and suppose a pressure exerted on the fluid at some part of the side $AE$ by a piston $a$; then this pressure will be transmitted through the fluid, not only in one direction, as would be the case with a rigid body, but in all directions around the piston. To test the truth of this, suppose a piston $b$, of the same size as $a$, to be inserted in the side $BC$, then it will be found that the same force will have to be applied to the piston $b$, to prevent its being thrust outward, as has been applied to the piston $a$ in order to produce the pressure on the surface of the fluid.




The same property may be proved by other experiments, so far as the nature of the case allows of experimental proof, and will be assumed as true in all that follows.

5. The pressure which a fluid exerts upon a smooth plane is necessarily perpendicular to the plane, because the pressure must be mutual, and a smooth plane is incapable of exerting any pressure parallel to its surface.

Thus for example, the pressure of the water upon the rudder of a ship acts in a direction perpendicular to the rudder, and thus has a tendency to turn the vessel out of its course unless the plane of the rudder is coincident with the vertical plane passing through the keel. Again the pressure of the air upon a sail, supposed plane, is perpendicular to the plane of the sail; consequently if the plane of the sail be so arranged as to lie between the point from which the wind blows and the point to which the vessel's course is directed, there will be a resolved part of the pressure on the sail in the direction of the intended course. This is the general principle upon which vessels are able to sail in a direction making an acute angle with that of the wind; the question how *near to the wind* a vessel can sail, depends upon practical considerations, the rig of the vessel, and the like.

When a plane surface is submitted to the pressure of a fluid the whole pressure is equivalent to one force perpendicular to the plane; but when a surface not plane, as that of a sphere or a cone for instance, is pressed by a fluid, the direction of the pressure will in general be different for all the different points of the surface, and the whole pressure on the surface will be the sum of an infinite number of pressures having different directions, and will not be a single force in a determinate direction.

6. Having spoken of the pressure on a plane, we must explain how such pressure is measured. The pressure may be either uniform or variable, that is, it may be the same at one point as at another, or it may vary from point to point: in the former case, the pressure is measured by the pressure produced on a unit of area: in the latter, the pressure at any point is measured by the pressure which would be produced on a unit of area, if the pressure at every point of it were the same as at the proposed point. The unit of area may be any whatever, as, for instance, 1 square inch. The pressure thus measured is called *the pressure referred to a unit of surface*, and is usually denoted by the letter $p$. Suppose, for example, the pressure on each square inch of the bottom of a pail of water were the same as would be produced by putting upon it a weight of 3 lbs., then $p = 3$. Also if $A$ be the whole area pressed, and the pressure be uniform, the whole pressure will be measured by $pA$: thus, in the example just now taken, if the bottom of the pail contain 40 square inches, $pA = 3 \times 40 = 120$ lbs., which is the whole pressure exerted by the water.

Conversely, if the whole pressure on an area be given, and the pressure be uniform, then the pressure referred to a unit of surface will be found by dividing the whole pressure by the quantity expressing the area.

When we speak of the pressure at any internal point of a mass of fluid, we mean the pressure which would be exerted supposing a rigid plane were made to pass through the point in question. Or, supposing a very small portion of fluid at the given point to become rigid, then we shall have the case of a small rigid body kept in equilibrium by equal pressures

on all sides of it, and the intensity of these pressures measures the pressure of the fluid at the proposed point.

7. Some very remarkable results follow from the law of equal transmission of fluid pressure, which at first, perhaps, appear somewhat paradoxical. For since, when we exert a pressure on the surface of a fluid, that pressure is transmitted equally in all directions, it is evident that the whole pressure produced on any surface will be proportional to the extent of the surface, and therefore may be increased indefinitely by increasing that surface. The following experiment exhibits this result from a very striking point of view. Suppose $AB$, $CD$ to be two boards forming the ends of an air-tight leather bag, and through the lower board $CD$ let a small tube, $EF$, be introduced; then it will be found, that, by making the board $AB$ sufficiently large, a person standing upon it and blowing into the tube will be able to lift his own weight with ease. This is sometimes spoken of as the *Hydrostatic paradox*.


8. It may be observed that the principle of Virtual Velocities is applicable to the case of equilibrium of an incompressible fluid under the action of no forces.

For let $V$ be the volume of a vessel containing incompressible fluid, and let $a_1 a_2 a_3 \ldots$ be the transverse sections of cylindrical pipes opening into the vessel: also let $p$ be the pressure referred to a unit of surface, which will be the same throughout the fluid. Suppose pistons to be fitted into the pipes, and let $x_1 x_2 x_3 \ldots$ be the distances of these respectively from the points in which the pipes meet the vessel.

Then the volume of the fluid

$$= V + a_1 x_1 + a_2 x_2 + a_3 x_3 + \ldots$$

Now suppose the pistons to be slightly displaced, some being moved toward the vessel, others drawn from it, and let $h_1 h_2 h_3 \ldots$ be the displacements, those being reckoned positive which take place from the vessel and those negative which take place towards it; then after the displacement, the volume of the fluid will be

$$V + a_1(x_1 + h_1) + a_2(x_2 + h_2) + a_3(x_3 + h_3) + \ldots$$

and therefore, by the condition of incompressibility, we must have

$$a_1 h_1 + a_2 h_2 + a_3 h_3 + \ldots = 0:$$

$$\therefore p a_1 h_1 + p a_2 h_2 + p a_3 h_3 + \ldots = 0.$$

But $pa_1$, $pa_2$, $pa_3$, ... are the pressures on the pistons, which if we denote by $P_1 P_2 P_3 \ldots$, the equation becomes

$$P_1 h_1 + P_2 h_2 + P_3 h_3 + \ldots = 0,$$

which expresses the condition given by the principle of Virtual Velocities. (See page 258.)

It is manifest that the principle is not applicable to compressible fluids.

## ON THE EQUILIBRIUM OF NON-ELASTIC FLUIDS UNDER THE ACTION OF GRAVITY.

9. In the treatise on Dynamics, (Art. 21, page 279), we explained what is meant by the mass of a body, and we established the formula

$$W = Mg,$$

where $W$ is the weight of a body, $M$ its mass, and $g$ the accelerating force of gravity (Art. 24, page 281).

By the *density* of a body we mean the quantity of matter contained in, i. e. the mass of, a unit of its volume; so that if $V$ be the volume of a body of uniform density $\rho$, and $M$ its mass, then

$$M = \rho V,$$

$$\text{and} \therefore W = \rho V g.$$

It will be observed that here, as in the case of mass, (see Dynamics, Art. 21, page 279) we are obliged to refer to the effect of gravity upon matter, and we consider two bodies of equal volume to be equally dense when their *weights* are equal, that is, when the effect of gravity upon them is the same.

The *specific* gravity of a body is the weight of a unit of its volume; so that if $S$ be the specific gravity of a body, the volume of which is $V$ and the weight $W$, then will

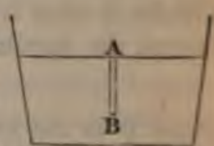
$$W = VS.$$

Comparing this with the formula last obtained, we see that

$$S = \rho g.$$

The specific gravities of different substances may be conveniently estimated with reference to some standard substance; for instance, distilled water at a given temperature; if we call the specific gravity of this standard substance 1, then those of other substances will be expressed by numbers, which give the ratios of the specific gravities of those substances to that of the standard. Thus, if the specific gravity of water is 1, that of lead is 11.35, of copper 8.9, and so of other substances.

10. *To find the pressure referred to a unit of surface at any depth below the surface of a fluid at rest.*




Let $B$ be a point at a depth $z$ below the surface; suppose $AB$ to be a prism of fluid of very small transverse section $a$, and suppose this prism to become solid, which may evidently be done without disturbing the equilibrium; then the pressure on the base of the prism will be its weight, or $\rho g a z$, if $\rho$ be the density of the fluid. Again, let $p$ be the pressure at $B$ referred to a unit of surface, then the whole pressure on the base of the prism $= pa$; hence we have

$$pa = \rho g a z,$$

$$\text{or } p = \rho g z.$$

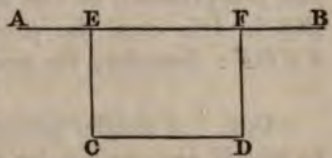
Hence the pressure at any point in the interior of a fluid at rest, is proportional to the depth below the surface.

If we suppose the surface of the fluid to be exposed to some pressure, as the pressure of the air, and we call this pressure $\Pi$, we shall have

$$p = \rho g z + \Pi.$$

11. The pressure which we have here determined is not in any definite direction, but exists in all directions around $B$ in virtue of the fundamental property of fluids. For instance, if we have a vessel with vertical sides containing fluid, then the pressure at a given depth on the sides of a vessel will be that which we have determined, but its direction will be horizontal.

12. *The surface of a fluid at rest is a horizontal plane.*




Let $AB$ be the surface of the fluid, $CD$ a horizontal plane below the surface, $E$, $F$ any two points in the surface, $EC$, $FD$ perpendicular to $CD$, $\rho$ the density of the fluid. Now, suppose a small canal of fluid joining $C$ and $D$, any two points in the given horizontal plane, to become a solid prism; then since this prism is in equilibrium, the horizontal pressures upon its two ends must be equal; but these are the fluid pressures at $C$ and $D$; hence

$$\text{fluid pressure at } C = \text{fluid pressure at } D,$$

$$\text{that is, } \rho g EC = \rho g FD;$$

$$\therefore EC = FD,$$

and therefore the surface of the fluid is parallel to the horizontal plane $CD$, or is horizontal.

13. *To find the pressure on a plane horizontal area at any depth below the surface of a fluid at rest.*

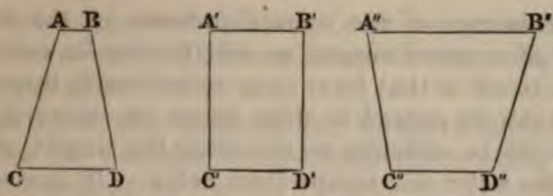
Suppose vertical lines be drawn from all points of the circumference of the plane area to the surface of the fluid,

and suppose the prism of fluid thus formed, having the given area for its base, to become solid; then the pressure on the plane area will be the same as before. But in this hypothetical case, the pressure manifestly equals the weight of the solid prism. Hence the pressure on the plane area is the weight of a column of fluid, the base of which is the area pressed, and height the depth of the area below the surface.

The proposition will be true, even when there is no such column of fluid actually superincumbent upon the plane. For, suppose we have a vessel of the shape *ACDB*, full of fluid; produce *AB* and draw *CE*, *DF* perpendicular to it, and suppose the part *ECA* to be filled with fluid; now let the side *AC* of the vessel be removed, then equilibrium will still subsist and the pressure on the base *CD* will be the same as before. But in this case the pressure is the weight of the column *ECDF*: therefore the proposition is still true.


COR. A similar proposition holds for any surface; that is to say, if any surface be submerged in a fluid the downward *vertical pressure* upon it is the weight of the column of fluid which is superincumbent upon the surface, or which would be superincumbent if the upper surface of the fluid be supposed extended, as in the second case of the preceding proposition. The proof of this proposition will be analogous to that which has been given above; but it is particularly to be noted that the pressure found in this case is the vertical pressure only, the amount of the whole pressure sustained by the surface will be investigated presently. (Art. 18.)

14. Hence it appears that the pressure on any plane horizontal area depends on its depth below the highest point of the fluid, and not upon the magnitude of the actual superincumbent mass. For instance, if we have three vessels, such as in the figure, having their bases and altitudes equal, the pressure on the bases when they are filled with fluid will be the same.



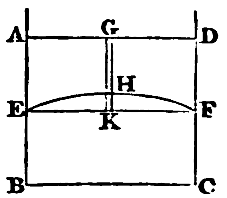

An illustration of the proposition is supplied also by such an experiment as the following; a barrel filled with water, and having a long vertical pipe of small transverse section introduced into it also filled with water, may be burst by the fluid pressure, if the pipe be of considerable length.

It may nevertheless appear strange, that it should be possible for fluid contained in a vessel to produce upon the base a pressure greater than its own weight. Consider, for instance, the vessel $ABCD$; if it is filled with water and placed upon a table, the pressure upon the table will be the weight of the vessel together with that of the water, and hence we might be disposed to conclude that the pressure on the base of the vessel must be equal to the weight of the water. The erroneous nature of this conclusion will however appear, if we consider that the water presses not only upon the base $CD$ of the vessel but also upon the sides; consider the pressure at any point of the line $AC$, it will be perpendicular to $AC$, and may therefore be resolved into two portions, one horizontal, and the other acting vertically upwards; the former will have no effect in determining the pressure on the base, but the latter tends to raise the vessel upwards, and must therefore be counteracted by the downward pressure of the fluid on the base of the vessel.

Hence we may say, that the fluid pressure on the base will be the weight of the fluid, together with a pressure sufficient to counteract the whole upward pressure of the fluid upon the sides of the vessel. In the case of a vessel shaped as $A''B''C''D''$ the pressure on the sides will be downwards, and therefore the pressure on the base must be the weight of the fluid diminished by the amount of the vertical pressure on the sides.

The existence of the upward pressure of the fluid upon the sides of a vessel, shaped as $ABCD$, may be exhibited by placing a vessel of that form open at both ends upon a table; then if water be poured in from above, the upward pressure will at length be sufficient to overcome the weight of the vessel, and the water will escape from below. If the weight of the vessel be given, and the form be supposed conical, it is not difficult, from what has been now said, to determine the height to which the water may rise so as just not to lift the vessel.

15. *The common surface of two fluids which do not mix is a horizontal plane.*




Let $ABCD$ be a vessel containing the fluids, and $AD$ the horizontal surface of the upper fluid. If possible, let $EHF$ be the common surface; draw the horizontal plane $EF$. Consider the equilibrium of a vertical column $GHK$, composed partly of one fluid and partly of the other; the pressure at $K =$ the weight of the column $GHK$, but the pressure at $K$ also equals the pressure at $E$ which is in the same horizontal plane with it, and therefore equals the weight of a column of fluid reaching from $E$ to the surface.

Hence the weight of a column composed of the two fluids equals that of a column of the same height composed of only one of them; which is absurd, since the fluids are supposed to be of different densities. Therefore the common surface cannot be as we have supposed, and must be horizontal.

16. Hence, theoretically, two fluids resting the one on the other will be in equilibrium, provided their common surface be horizontal; but practically equilibrium will not subsist, unless the lower fluid be that of greater density; for if the contrary were the case, the smallest disturbance of the fluids would cause the denser fluid to descend, and the equilibrium would be destroyed: in the former case the equilibrium is said to be *stable*, in the latter *unstable*. Thus oil can rest permanently on water, but not *vice versâ*.

17. *When two fluids meet in a bent tube, the altitudes of their surfaces above the horizontal plane in which they meet are inversely as their densities.*

For let $\rho\rho'$ be the densities, and $zz'$ the altitudes of the fluids above the common surface, then the pressure referred to a unit of surface of the two fluids at the common surface must be equal and opposite, because there is equilibrium; call it $p$; then, considering the first fluid, we have (Art. 10)

$$p = \rho g z.$$

Considering the second, we have

$$p = \rho' g z';$$

$$\therefore \rho z = \rho' z',$$

$$\text{or } \frac{z}{z'} = \frac{\rho'}{\rho}.$$

18. *The whole fluid pressure on a surface immersed in a fluid is equal to the weight of a column of fluid, having for its base the area of the surface immersed, and for its height the depth of the centre of gravity of the surface below the surface of the fluid.*

Suppose the surface divided into a number of very small portions, each of which we may consider to be ultimately plane, and to have all its points at the same distance below the surface of the fluid. Let $a_1, a_2, a_3$...... be the areas of the small portions, and $z_1, z_2, z_3$ ...... their respective depths below the surface, then the pressure on $a_1$ is $\rho g a_1 z_1$, on $a_2$ the pressure is $\rho g a_2 z_2$, and so on; hence the whole pressure on the surface

$$= \rho g\, (a_1 z_1 + a_2 z_2 + a_3 z_3 + \ldots\ldots).$$

Let $S$ be the area of the surface, and $z$ the depth of its centre of gravity, then by a property of the centre of gravity*, (see Statics, Art. 44, page 250,) we have,

* The property referred to was proved for a number of heavy particles; when we apply the same to the centre of gravity of a surface, we must suppose it to be a physical surface of an indefinitely small thickness and having weight; it may then be considered to be made up of the component particles, the weights of which are proportional to $a_1, a_2$......

$$\bar{z} = \frac{a_1 z_1 + a_2 z_2 + a_3 z_3 + \ldots\ldots}{a_1 + a_2 + a_3 + \ldots\ldots},$$

$$= \frac{a_1 z_1 + a_2 z_2 + a_3 z_3 + \ldots\ldots}{S};$$

$\therefore$ the whole pressure on the surface $= \rho g S \bar{z}$,

$=$ the weight of a column of fluid of base $S$ and height $\bar{z}$.

Ex. 1. An isosceles triangle is immersed in fluid, having its vertex in the surface of the fluid, and its base horizontal; find the whole pressure on the plane of the triangle.

Let the base of the triangle $= a$,

the perpendicular from the vertex on the base $= b$,

the angle at which the plane of the triangle is inclined to the horizon $= \theta$;

$\therefore$ the depth of the centre of gravity $= \dfrac{2b}{3} \sin\theta$.

(Statics, Art. 46, page 251.)

Also the area of the triangle $= \dfrac{ab}{2}$;

$\therefore$ the whole pressure $= \dfrac{1}{3} \rho g a b^2 \sin\theta$.

Ex. 2. A cylindrical vessel, having its axis vertical, is full of fluid; find the whole pressure on the sides.

Let $h$ be the height of the vessel, $r$ the radius of the base; then the surface pressed $= 2\pi r h$, and the depth of the centre of gravity $= \dfrac{h}{2}$; $\therefore$ the whole pressure $= \pi \rho g h^2 r$.

19. The pressure on a surface, which we have been considering, is not a single pressure in a certain direction, nor does it admit, in general, of a single resultant, because the direction of the pressure on any one of the small areas into which we have supposed the surface to be divided is perpendicular to that small area, and therefore varies from point to

point of the surface, except in the case of a plane area. If, however, we consider only that portion of the fluid pressure which acts in any given direction, we may determine the single force in that direction, to which all the fluid pressures at different points of the surface are equivalent.

20. *When a body is immersed in a heavy fluid, the resultant of the horizontal pressures at all points of the surface of the body is zero.*

The pressure on the surface of the body will be the same in every respect as on a similar and equal portion of the fluid, supposed to be substituted for the body, and then made solid. And this hypothetical solid will be in equilibrium under the action of its own weight, and the pressure of the fluid; but no part of its own weight acts horizontally, therefore the horizontal part of the fluid pressure must be zero.

*Under the same circumstances, the resultant of the vertical pressure on the body is equal to the weight, and acts through the centre of gravity of the fluid displaced.*

Making use of the same artifice as before, the portion of fluid supposed to become solid is kept in equilibrium by its own weight and the vertical pressure of the fluid, and these must be equal and opposite forces; but the former may be supposed to act at the centre of gravity of the solidified portion, i. e. of the fluid displaced; therefore also the vertical pressure of the fluid is equal to the weight of that solidified portion, and acts through its centre of gravity.

21. *To determine the conditions of equilibrium of a floating body.*

The floating body is kept in equilibrium by its own weight acting downwards through its centre of gravity, and the pressure of the fluid acting upwards, which, as we have shewn, is equal to the weight, and acts through the centre of gravity, of the fluid displaced. Hence, when a body floats in equilibrium, the weight of the body is equal to that of the fluid displaced,

and the centres of gravity of the body and of the fluid displaced are in the same vertical line*.

Cor. If a body is wholly immersed in a fluid of greater specific gravity than itself, and is prevented from rising by a string or otherwise, then the force tending to raise the body is the difference between its own weight and that of the fluid displaced.

Let $V$ be the volume of the body, $S$ its specific gravity, $S'$ that of the fluid;

$$\text{then the pressure of the fluid upwards} = VS',$$

$$\text{weight of the body acting downwards} = VS;$$

$$\therefore \text{force tending upwards} = V(S' - S).$$

Hence we see the reason of the ascent of a balloon, when inflated with a gas specifically lighter than common air.

The balloon first employed for making aerial ascents consisted of a light spherical envelope having a circular aperture, under which a fire having been made, the air within the balloon became heated, and therefore specifically lighter than the external air. The condition necessary for the ascent of the balloon was thus satisfied, and ascents were actually made by persons in machines upon this construction; at present this kind of balloon is used merely as a toy, the plan now adopted being much more safe and admitting of more complete control. The balloon now used consists of an envelope which is inflated with a gas specifically lighter than air, and from which is suspended a car for the conveyance of the aeronaut; the balloon is furnished with a valve at its summit, which is kept carefully closed by a spring, but which can be opened by means of a string which is within reach of the person in the car. The car also carries a quantity of fine sand for ballast. When the balloon first ascends, it is usual not to fill it entirely with gas; as it rises however and the atmospheric air becomes

* To find the positions in which a given solid will float in a fluid is very difficult as a matter of mathematical calculation, even in cases apparently simple. The problem is evidently merely geometrical, and may be enunciated thus, To divide a solid by a plane into two parts, such that their volumes shall be in a given ratio and the line joining their centres of gravity perpendicular to the cutting plane.

less dense, the gas dilates, and completely fills the balloon; as the balloon continues to ascend, the external pressure becomes less, and there would be danger of the pressure of the gas within causing it to burst, if it were not for the escape of the gas which is permitted by means of the valve already mentioned. The balloon ascends more and more slowly as it rises into regions in which the air is less and less dense, and at length it would completely stop. If the aeronaut desires to ascend beyond this limit, he throws ballast out of the balloon, and thus diminishes its weight. When he wishes to descend he opens the valve and allows the gas gradually to escape. The problem of the ascent and descent of a balloon has been thus completely solved: that of guiding its course according to the wish of the aeronaut has been frequently attempted, but hitherto without success.

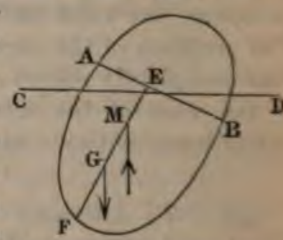
22. *To determine whether the equilibrium of a floating body is stable or unstable.*

Suppose the floating body to be slightly displaced from its position of equilibrium, by being made to revolve through a very small angle in a vertical plane; then a new fluid pressure will be called into action, which will in general not act through the centre of gravity of the solid. If the tendency of the fluid pressure be to bring the body back again to its equilibrium position the equilibrium is stable; if otherwise, unstable. It is evidently necessary for the absolute stability of the equilibrium, that the equilibrium should be stable for a displacement in *any* vertical plane; or if we can ascertain from general considerations, in any particular case, the plane for which the tendency after displacement to return to the position of equilibrium is least, and can assure ourselves that the equilibrium is stable for that plane, then we may conclude that the equilibrium is absolutely stable. For example, if the equilibrium of a ship be stable for disturbances in the plane perpendicular to its length the equilibrium will be altogether stable.

The determination of the mathematical condition of stability requires a higher calculus than is introduced into

this work, but the nature of the process may be easily explained. For simplicity's sake we shall suppose the body to be symmetrical about the plane in which the displacement takes place, which we shall suppose to be the plane of the paper; also we shall suppose the displacement to take place subject to the condition, that the quantity of fluid displaced before and after the disturbance is the same.

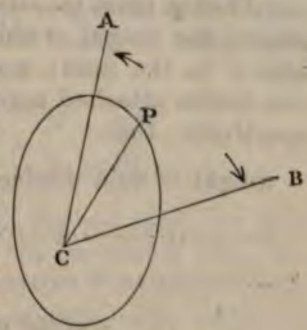
Let $CD$ be the surface of the fluid, $AB$ the section of the body which in the position of equilibrium coincided with the surface of the fluid, that is, the plane of floatation, $EGF$ the line which was vertical and which contained the centres of gravity of the body ($G$) and that of the fluid displaced. Then after the disturbance the body will be acted upon by two forces, namely, its own weight vertically downwards through $G$ and an equal force acting vertically upwards through the centre of gravity of the fluid displaced; and on account of the supposed symmetry of the body about the plane of disturbance the direction of this latter force will lie in that plane and will therefore intersect $EGF$ in some point, as $M$. $M$ is called the *Metacentre* of the body, and its position may be calculated mathematically; if $M$ be above $G$ as in the figure, it is evident that the forces tend to bring back the body to its position of equilibrium; if, on the other hand, $M$ the lower than $G$, the reverse is the case; consequently we may say, that the equilibrium is stable or unstable according as the centre of gravity of the body is lower or higher than the metacentre.




23. It may be observed that if the body be made to revolve, subject to the condition of the quantity of fluid displaced being always the same, it will assume successively positions of *stable* and *unstable* equilibrium.

For let $CP$ be a fixed line in the body, and suppose that when $CP$ coincides with the fixed line $CA$ the body is in a position of stable equilibrium, and that when the same line $CP$ coincides with another fixed line $CB$ the body is also in a

position of stable equilibrium; and suppose that there is no position of stable equilibrium corresponding to any position of $CP$ between $CA$ and $CB$. Then if $CP$ be brought near to $CA$ it tends to coincide with $CA$, by the nature of stable equilibrium; and if it be brought near $CB$ it tends to coincide with $CB$; consequently there must be a position for $CP$ between $CA$ and $CB$ for which it will tend to move neither towards $CA$ nor $CB$, that is, there must be a position or equilibrium. And we have supposed that there is no position of *stable* equilibrium, consequently between each two positions of stable equilibrium there is one of *unstable*. And there cannot be more than one, because similar reasoning would shew that between each two positions of unstable equilibrium there is one of stable. Hence the proposition enunciated is true.




## ON FINDING THE SPECIFIC GRAVITY OF A SUBSTANCE.

24. We have before remarked (Art. 9) that it is convenient to compare the specific gravities of substances with that of some standard substance, such as distilled water at a given temperature; we shall now explain the mode of making the comparison.

We will first consider the case of a solid body.

(1) Let the body be of greater specific gravity than the fluid.

Weigh the body first in vacuum, and then in the fluid, and let the weight in the first case be $W$, and in the second $W'$; then the weight of the fluid displaced $= W - W'$;

$$\therefore \frac{\text{specific gravity of solid}}{\text{.................... fluid}} = \frac{W^*}{W - W'}.$$

* In this manner also the specific gravity of two solids may be compared with each other; let us take a portion of a second solid, and suppose that its weight in vacuum is $W$

(2) Let the body be of less specific gravity than the fluid.

Then it must be attached to a piece of some heavy substance, the weight of which we will suppose to be $w$ in vacuum, and $w'$ in the fluid; and let $W_1$ $W'_1$ be the weights of the two bodies attached together, in vacuum and in the fluid respectively; then

weight of fluid displaced by the two $= W_1 - W'_1$.

.................................. heavier $= w - w'$,

∴ .................................. lighter $= W_1 - W'_1 - w + w'$;

$$\text{and } \therefore \frac{\text{specific gravity of solid}}{\text{......................... fluid}} = \frac{W}{W_1 - W'_1 - w + w'}.$$

If the body be composed of a substance soluble in the fluid, we must inclose it in wax and proceed as before.

In rough experiments, founded on the preceding investigation, it will be sufficient to weigh the bodies in *air* instead of in vacuum; but in all delicate experiments, the weight of the air displaced by the body must be added to its apparent weight in air.

25. *To determine the specific gravity of air.*

Let a large flask be filled with air, and weighed, and let

as in the case of the first, and its weight in the fluid $W'''$; then we shall have

$$\frac{\text{specific gravity of first solid}}{\text{specific gravity of second solid}} = \frac{W - W'''}{W - W'}.$$

The very simple method applied by Archimedes to solve this problem in the well-known case of the golden crown deserves notice. The story is as follows: Hiero weighed out a certain portion of gold to a workman, which was to be constructed into a crown; the crown was made, and the weight found correct; there was a suspicion however that some baser metal had been mixed with the gold, and Archimedes was directed to endeavour to detect the cheat. This he did by weighing out a quantity of gold and a quantity of silver, each equal in weight to the crown; he then filled a vessel with water, and immersing the gold observed the quantity of water displaced; he repeated the operation with the silver; and lastly with the crown. When he compared the quantities of water displaced in the three experiments, he found that that displaced by the crown was intermediate in amount to the other two. Hence it appeared that some other metal had been mixed with the gold; and assuming that metal to be silver, Archimedes was able to calculate the exact proportion of gold and silver contained in the crown.

the weight be $W$; again, let the air be exhausted, and the flask weighed, and its weight be $W'$; lastly, let the flask be filled with water, and weighed, and its weight be $W''$. Then the weight of air contained is $W - W'$. and of water contained $W'' - W'$,

$$\therefore \frac{\text{specific gravity of air}}{\text{.................. water}} = \frac{W - W'}{W'' - W'}.$$

The specific gravity of dry air will depend upon the temperature and also upon the state of the barometer (Art. 32); when the barometer stands at 30 inches and Fahrenheit's thermometer (Art. 44) at 60°, the weight of 1000 cubic inches of dry air is about 310 grains.

26. The apparent weight of a body, resulting from an experiment made in common air, is always deceitful, except in the case of the substance weighed being of the same material as the weights used in the opposite scale of the balance.

Let $V$ be the volume of the body, $S$ its specific gravity,

$V'$ ............ the weight, $S'$ .....................

$\sigma$ the specific gravity of air.

Then we must have,

$$V(S - \sigma) = V'(S' - \sigma);$$

$$\therefore VS = V'S' \frac{1 - \dfrac{\sigma}{S'}}{1 - \dfrac{\sigma}{S}}.$$

Hence the apparent weight of a body must be multiplied by the factor $\dfrac{1 - \dfrac{\sigma}{S'}}{1 - \dfrac{\sigma}{S}}$ in order to get the true weight.

27. *Given volumes of substances of known specific gravities are compounded; to find the specific gravity of the compound.*

Let $V$ $V'$ be the volumes,

$S$ $S'$ the specific gravities,

$\sigma$ the specific gravity of the compound.

Then, since the weight of the compound equals the sum of the weights of the constituents, we have

$$(V + V')\,\sigma = VS + V'S';$$

$$\therefore \sigma = \frac{VS + V'S'}{V + V'}.$$

Obs. It is here assumed that the volume of the compound is equal to the sum of the volumes of the constituent fluids; an assumption not always strictly true.

28. *To compare the specific gravities of two fluids by weighing the same solid in each.*

Let

$W$ be the weight of the solid in the vacuum,

$W_1$ its apparent weight when suspended in the first fluid,

$W_2$............................................ second.

Then,

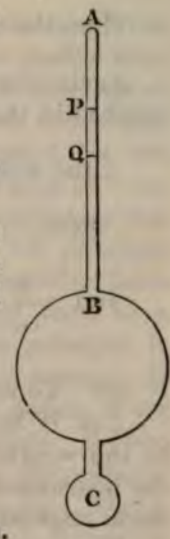
weight of the quantity of the first fluid displaced $= W - W_1$,

................................second ............ $= W - W_2$;

$$\therefore \text{ the ratio of the specific gravities} = \frac{W - W_1}{W - W_2}.$$

29. *The specific gravities of two fluids may be conveniently compared by means of the common hydrometer.*

This instrument consists of two hollow spheres, $B$ and $C$, having their centres in the axis of the graduated stem $AB$; the sphere $C$ is loaded with lead or mercury, so that the instrument will float in a fluid with the stem vertical.




Let $S\,S'$ be the specific gravities of two fluids which are to be compared,

$V$ the volume of the instrument,

$W$ its weight,

$k$ the area of the transverse section of the stem;

and suppose that when the instrument is made to float in the two fluids, the level of the fluid in the first case is $P$, and in the second $Q$; then

$$W = S(V - k\,.\,AP),$$

$$\text{also } W = S'(V - k\,.\,AQ);$$

$$\therefore \quad \frac{S}{S'} = \frac{V - k\,.\,AQ}{V - k\,.\,AP}.$$

Hence by measuring $AP$, $AQ$, the ratio $\frac{S}{S'}$ is known.

In practice the hydrometer is so graduated, that the specific gravity of any fluid into which it is plunged, as compared with that of water, can be ascertained by inspection.

30. *Nicholson's Hydrometer.*

This is a convenient instrument for comparing either the specific gravities of a solid and a fluid, or the specific gravities of two fluids.

$AB$ is a hollow cylinder; $C$ a dish supported by a wire $AC$ coinciding with the axis of $AB$; $D$ another dish suspended from the lower extremity of $AB$.

(1) To compare the specific gravities of a solid and a fluid.

Let $W_1$ be the weight, which placed in $C$ causes the instrument to sink in the fluid till the surface of the fluid meets $AC$ in a given point $E$. Place the solid in $C$ and let $W_2$ be the weight which must be added to make the instrument sink as deep as before. Place the solid in $D$, and let $W_3$ be the weight which must then be placed in $C$ in order to sink the instrument to the same depth.


Then the weight of the solid $= W_1 - W_2$.

Again, the apparent weight of the solid when weighed in the fluid $= W_1 - W_3$;

$\therefore$ the weight of the fluid displaced

$$= (W_1 - W_2) - (W_1 - W_3) = W_3 - W_2;$$

$$\text{and } \therefore \frac{\text{specific gravity of solid}}{\text{.................... fluid}} = \frac{W_1 - W_2}{W_3 - W_2}.$$

(2) To compare the specific gravities of two fluids.

Let $W$ be the weight of the hydrometer; and let $W_1$ $W_2$ be the weights which must be placed in $C$ in order to sink the instrument down to the point $E$, when floating in the two fluids respectively.

The weight of the fluid displaced in the two cases will be $W + W_1$ and $W + W_2$; but the volume displaced is the same;

$$\therefore \text{the ratio of the specific gravities} = \frac{W + W_1}{W + W_2}.$$

## ON THE PRESSURE OF AIR AND OTHER ELASTIC FLUIDS.

31. The atmosphere or air, which surrounds the earth, produces a pressure upon all bodies immersed in it. This pressure, though very great, is not in general felt by us, because by the nature of fluid pressure it acts equally on all sides of a body submitted to it; for instance, when a man raises his hand, the downward pressure of the air above his hand is equal to the upward pressure below it, and the two therefore neutralize each other.

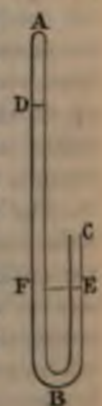
If however the pressure of the air were allowed to act upon one side of a body, and not on the other, the effect would be immediately sensible; thus if the air within a cup be rarefied by heat and placed upon the hand, it will adhere to it, owing to the pressure of the ambient air. And so if two brass hemispheres be made to fit accurately, and the air from within the sphere be withdrawn, a violent effort is required

to separate the hemispheres; thus if the diameter of the sphere be 14 inches, the force required is about half a ton; this experiment was made by Otto Guericke, the inventor of the air-pump. Effects of this kind had forced themselves upon the attention of the ancients, who invented for the purpose of accounting for them the famous principle of "nature's abhorrence of a vacuum:" a principle concerning which Dr Whewell remarks, "we must contend that the principle was a very good one, inasmuch as it brought together facts which are really of the same kind, and referred them to a common cause. But when urged as an ultimate principle, it was not only unphilosophical, but imperfect and wrong. It was unphilosophical, because it introduced the notion of an emotion, horror, as an account of physical facts: it was imperfect, because it was at best only a law of phenomena, not pointing out any physical cause; and it was wrong, because it gave an unlimited extent to the effect." Torricelli in 1643 produced the equilibrium between the weight of a column of mercury and the weight of the atmosphere, which constitutes the principle of the barometer about to be described immediately; and the correctness of the explanation of the phenomena was put beyond a doubt by the experiment of Pascal, who suggested that if the support of the column of mercury were due entirely to the weight of the atmosphere, then a less column would be supported at the top of a hill than at the bottom; this famous experiment was made upon the Puy de Dome in Auvergne in 1648 with complete success; it was found also that a bladder, partly filled with air and carefully closed, on being carried to the top of the mountain expanded by the dilatation of the air within, and that the reverse took place when it was again brought to the foot of the mountain.

The atmosphere forms only a thin coating upon the surface of the earth; the exact height to which it extends is not easily ascertained, but that it must be limited will be seen at once from the fact that if it revolve with the earth, (assuming the earth to be a body revolving about its axis, as will be explained hereafter,) each particle will be acted upon by two forces, the attraction of the earth and the centrifugal

force due to the rotation; now the attraction of the earth diminishes in ascending from its surface and the centrifugal force increases, consequently there will be a limit beyond which the centrifugal force will render the existence of an atmosphere revolving with the earth impossible. This condition however gives a limit of about 22000 miles, which is undoubtedly far beyond the truth; it seems probable that there is no sensible atmosphere beyond a height of about 50 miles. That the limit of the atmosphere is not greater than this seems probable from considerations of the decrease of temperature in ascending above the earth's surface; for judging from the rate of decrease throughout that portion of the atmosphere in which observations have been made, it would seem probable that at a height of 50 miles above the earth's surface the temperature is such, as to render it impossible for atmospheric air to retain its gaseous form. Supposing the atmosphere then to have a height of something like 50 miles, and observing that the earth may be spoken of approximately as a sphere having a radius of 4000 miles (see Astronomy, Art. 7), it will appear that the atmosphere forms a comparatively thin envelope upon its surface; if we conceive for example of the earth as a globe of one foot in diameter, the thickness of the atmosphere would be about the thirteenth part of an inch in thickness.

32. *To measure the pressure of the air.* (The Barometer.)

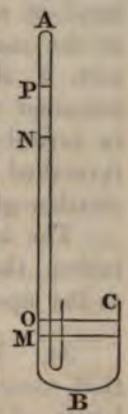



Let a bent glass tube $ABC$ be closed at the end $A$, and let $AB$ be filled with mercury. Then if the tube be placed so that $AB$ is vertical, the mercury will descend in $AB$ and rise in $BC$, leaving a vacuum above the level of the mercury in $AB$. Let $D$, $E$ be the levels of the mercury in the two branches, and draw $FE$ horizontal through $E$. Then the column of mercury $FD$ is supported by the pressure of the air on the surface at $E$, and therefore if $\Pi$ be the atmospheric pressure referred to a unit of surface, $\sigma$ the specific gravity of mercury, and $FD = h$, we shall have

$$\Pi = h\sigma.$$

This barometer is known as the *Siphon Barometer.* (Art. 49).

33. The barometer in common use differs slightly from the instrument just described.




The common barometer consists of a vertical closed tube $AB$ opening into a vessel $BC$; a scale of inches is attached to $AB$. The height of the mercurial column, as shewn by such an instrument as this, is the height above a fixed horizontal plane, not above the level of the mercury in $BC$ which is variable; hence the height will be in error, but since the area of the vessel $BC$ is much greater than that of the tube $AB$ the error will not be very great.

The actual error may easily be calculated, thus: let $O$ be the zero point of graduation, and when the mercury in $BC$ stands at that level, let $N$ be the level of the mercury in $AB$; and when the mercury in $AB$ has risen to $P$, let that in $BC$ have fallen to $M$, then $OM$ is the error required. Let $k$ $K$ be the areas of the transverse sections of $AB$ and $BC$ respectively;

$$\therefore k \,.\, PN = K \,.\, OM;$$

$$\therefore OM = \frac{k}{K} \,.\, PN = \frac{k}{K}(OP - ON);$$

and the true height of the barometer $= OP\left(1 + \frac{k}{K}\right) - \frac{k}{K} ON.$

There are several variations in the construction of the barometer; there is one very convenient for purposes requiring the transport of the instrument, in which the pressure of the air is admitted upon the surface of the mercury through an aperture in the glass tube so small as not to allow of the escape of the mercury; there is another in which the change of level of the mercury is obviated by making the bottom of the vessel containing it moveable by a screw, so that before making an observation the instrument can be so adjusted, that the surface of the mercury shall be on a level with the zero point of graduation. The *wheel* barometer may also deserve a passing notice; in this construc-

tion, which is available only for the purposes of a weather-glass and not as a scientific instrument, two weights are connected by a string which passes over an horizontal axis carrying a pointer which moves in a vertical plane like the hand of a clock; one of the weights rests upon the surface of the mercury in a siphon barometer, and rises and falls with it, the other weight which nearly counterpoises it is intended to stretch the string over the axis and cause it to turn by the friction between them. The pointer being furnished with a dial-plate the construction of the ordinary weather-glass is complete*.

The height of the barometer varies from about 28 to 31 inches, the average pressure of the atmosphere is about 15 lbs. upon a square inch.

34. We remarked in the commencement of this treatise, that some fluids were elastic and some non-elastic; in the latter, of which we have hitherto principally treated, the density is the same to whatever pressure the fluid may be subjected; but in elastic fluids the volume is diminished by pressure, and consequently the density increased. There will be, therefore, some relation between the volume occupied by an elastic fluid, and the pressure exerted by it in consequence of its elasticity.

*The pressure of air at a given temperature varies inversely as the space it occupies.*

We shall shew how this is proved experimentally, (1) when the air is compressed for the experiment, (2) when it is rarefied.

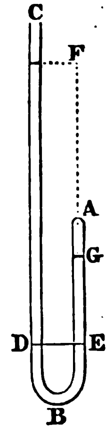
(1) Let $ABC$, a bent glass tube closed at $A$, and having its branches parallel, be placed so that the axes of the tube are vertical.

Pour mercury into the tube and by withdrawing some of

* Of late years a barometer of an entirely different construction from that described in the text has come into very general use. This is the *aneroid barometer*. Its construction consists essentially of a closed vessel, from which the air has been exhausted, and which is composed of material sufficiently elastic to yield sensibly to the varying pressure of the atmosphere. It admits of considerable delicacy, and is very convenient on account of its thoroughly portable character.

the air in $AB$, or by other means, make it stand at the same height in the two branches, at the level $DE$ suppose.

Now pour in more mercury, until the level in the two branches is $F$ and $G$ respectively.




Then if the ratio of the spaces $AE$, $AG$, occupied by the air in the two cases, be ascertained by weighing the mercury they will contain, and if $h$ be the height of the barometer at the time of the experiment, it will be found that

$$\frac{h+FG}{h} = \frac{\text{volume } AE}{\text{volume } AG}.$$

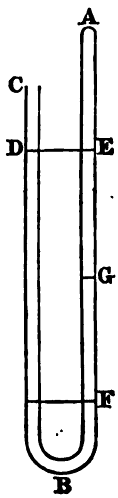
But if $\Pi$, $\Pi'$ be the pressure of the air when occupying the spaces $AE$, $AG$ respectively, and $\sigma$ the specific gravity of mercury,

$$\text{then } \Pi = \sigma h,$$

$$\Pi' = \sigma(h + FG);$$

$$\therefore \frac{\Pi'}{\Pi} = \frac{\text{volume } AE}{\text{volume } AG}.$$

(2) Let a glass tube $ABC$, closed at $A$, and having the branches $AB$, $BC$ parallel and nearly equal, be placed so that the axes of the branches are vertical.




Pour mercury into the tube, and make the surfaces in the two branches stand at the same height $DE$, as before.

Withdraw a portion of the mercury, and let the surface in the two branches then stand at $F$ and $G$ in the branches $BC$, $AB$ respectively.

Then it is found, as in the former case, that

$$\frac{h-FG}{h} = \frac{\text{volume } AE}{\text{volume } AG}.$$

But if $\Pi$, $\Pi'$ be the pressure of the air when occupying the spaces $AE$, $AG$ respectively, we shall have

$$\Pi = \sigma h,$$

$$\Pi' = \sigma(h - FG);$$

$$\therefore \frac{\Pi'}{\Pi} = \frac{\text{volume } AE}{\text{volume } AG}.$$

The elastic force acquired by air in an extreme state of compression is very great, and may be applied to a variety of purposes. The air-gun is an example of such application; in this case we have a quantity of air condensed in a small strong vessel, from which it can be permitted to escape suddenly by means of a valve; the air escapes into a small tube; in this a bullet is placed, which the elastic force is sufficient to expel with considerable force. The elastic force of compressed air is also usefully employed in the case of the fire-engine, as we shall see hereafter.

Under great pressure and extreme cold combined, and in some cases by the latter means only, a great number of elastic fluids, or gases, have been reduced from the elastic to a liquid or even solid state. Atmospheric air has however not been so transformed at present.

Cor. If $p$ be the pressure of air referred to a unit of surface, when the density is $\rho$, we have

$$p \propto \frac{1}{\text{volume}},$$

$$\text{but } \rho \propto \frac{1}{\text{volume}};$$

$$\therefore p \propto \rho = k\rho, \text{ suppose,}$$

where $k$ is a constant quantity.

The best observations give

$$\sqrt{k} = 916.2724 \text{ feet}.$$

The same law is found to hold good in the case of all elastic fluids.

35. *If the atmosphere be supposed to be divided into indefinitely thin strata of equal thickness, the densities of the air in those strata will be in geometrical progression.*

Suppose the strata to be so thin that the density may be supposed the same throughout each; and let $\rho_n$, $p_n$ be the density and pressure in the $n^{\text{th}}$ stratum measured from the earth's surface; $\tau$ the thickness of the strata.

Then the difference of pressure in passing from the $n^{\text{th}}$ to the $\overline{n+1}|^{\text{th}}$ stratum is the weight of a column of air of height $\tau$;

$$\therefore\ p_n - p_{n+1} = \rho_n g\tau \quad \text{(Art. 10)};$$

$$\text{but } p_n = k\rho_n \quad \text{(Art. 34)},$$

$$\text{and } p_{n+1} = k\rho_{n+1};$$

$$\therefore\ \frac{\rho_n - \rho_{n+1}}{\rho_n} = \frac{g\tau}{k};$$

in like manner, $$\frac{\rho_{n-1} - \rho_n}{\rho_{n-1}} = \frac{g\tau}{k};$$

$$\therefore\ \frac{\rho_n - \rho_{n+1}}{\rho_n} = \frac{\rho_{n-1} - \rho_n}{\rho_{n-1}},$$

$$\text{or } \rho_{n-1}\rho_{n+1} = \rho_n^2.$$

Hence the densities $\rho_1, \rho_2, \rho_3 \ldots$, and therefore also the pressures, $p_1, p_2, p_3 \ldots$, form a geometrical progression.

Obs. The preceding proposition is not experimentally true, for two reasons; *first*, we have considered the temperature to be the same at all heights above the earth's surface, which is not the case; and, *secondly*, we have neglected to take account of the diminution of the force of gravity as we recede from the centre of the earth. For small heights, however, the proposition may be taken as approximately true.

36. *To explain the method of finding the difference of altitude of two stations above the earth's surface by means of the barometer.*

Let $x$ be height in feet of one station above the earth's surface,
$x'$ ...................... the other ...............................

We may suppose the atmosphere to consist of strata of one foot thick, throughout each of which the pressure is the same, but that in passing from one to another of them the pressure diminishes in a geometrical progression. Let $r$ be the ratio of this progression, then (making $\tau = 1$ in the last article),

$$1 - r = \frac{g}{k}; \qquad \therefore\ r = 1 - \frac{g}{k}.$$

Again, let the height of the barometer at the two stations be $h\,h'$, which will be proportional to the atmospheric pressures at the two stations;

$$\therefore \frac{h}{h'} = \frac{r^x}{r^{x'}} = \left(1 - \frac{g}{k}\right)^{x-x'};$$

taking logarithms, $\log \dfrac{h}{h'} = (x - x') \log \left(1 - \dfrac{g}{k}\right)$,

$$\text{or } x - x' = \frac{\log \dfrac{h}{h'}}{\log \left(1 - \dfrac{g}{k}\right)};$$

which formula, by the aid of a table of logarithms, will give us the difference of height of the two stations, measured in feet*.

Obs. The preceding investigation explains the principles

* The student who is acquainted with the exponential theorem (see Ex. 5, p. 95) may solve this problem completely, as follows:

Let the thickness of the strata be $\tau$, and let $m\tau$, $n\tau$ be the heights of the two stations, and $x$ the difference of their heights, so that

$$x = (m - n)\,\tau,$$

then $\dfrac{h}{h'} = (1 - \dfrac{g}{k}\tau)^{m-n} = (1 - \dfrac{g}{k}\tau)^{\frac{x}{\tau}}$

$$= 1 - \frac{x}{\tau}\frac{g}{k}\tau + \frac{\frac{x}{\tau}\left(\frac{x}{\tau} - 1\right)}{1\,.\,2}\left(\frac{g}{k}\tau\right)^2 - \ldots\ldots \text{ by the binomial theorem,}$$

$$= 1 - x\frac{g}{k} + \frac{x(x-\tau)}{1\,.\,2}\left(\frac{g}{k}\right)^2 - \ldots\ldots$$

Now make $\tau = 0$, i.e. suppose the strata to be indefinitely thin;

$$\therefore \frac{h}{h'} = 1 - x\frac{g}{k} + \frac{1}{1\,.\,2}\left(x\frac{g}{k}\right)^2 - \ldots\ldots$$

$$= e^{-x\frac{g}{k}} \text{ (where } e = 2.7182818\ldots\ldots) \text{ by the exponential theorem;}$$

taking logarithms,

$$\log \frac{h}{h'} = -x\frac{g}{k}\log e,$$

$$\text{or } x = \frac{k}{g}\,\frac{\log \frac{h'}{h}}{\log e}.$$

of finding heights by barometrical observations, but requires many corrections in practice to enable us to obtain accurate results. The decrease of the earth's attraction, and the change of temperature, in ascending above the earth's surface, give rise to the two most important corrections. It will also be necessary to have regard to the pressure of the vapour which is held in solution by the air, and which will cause the observed height of the barometer to be an erroneous indication of the pressure of the atmosphere.

Even taking every precaution too much reliance must not be placed upon barometric determination of heights, except as a differential process for stations at no great distance from each other. For barometric measurements, as Sir J. Herschel observes, "rely in their application on the assumption of a state of equilibrium in the atmospheric strata over the whole globe, which is very far from being their actual state. Winds, especially steady and general currents sweeping over extensive continents, undoubtedly *tend* to produce some degree of conformity in the curvature of these strata to the *general* form of the land surface, and therefore to give an undue elevation to the mercurial column at some points. On the other hand, the existence of localities on the earth's surface where a permanent depression of the barometer prevails to the astonishing extent of nearly an inch, has been clearly proved by the observations of Ermann in Siberia, and of Ross in the Antarctic Seas, and is probably a result of the same cause, and may be conceived as complementary to an undue habitual elevation in other regions."

In default of a barometer, the pressure of the atmosphere may be determined by observing the temperature at which water will boil. For it is well known, that the temperature at which water will boil depends upon the pressure of the atmosphere, or upon the height of the barometer (Art. 46); and conversely, the height of the barometer may be inferred from the observed temperature of boiling water.

The following is the method of computing heights by means of the barometer, put in a convenient practical form, but not involving all corrections which might be applied: it is given in this form by Hutton.

The formula 10000 log $\frac{h'}{h}$ will give the altitude in fathoms, in the mean temperature of $31^{0}$ Fahrenheit (Art. 46); and for every degree of the thermometer above that, the result must be increased by so many times its $435^{\text{th}}$ part; $h'$ being the height of the barometer at the lower station, and $h$ at the higher. The practical method of applying this formula may be expressed in words by the following rules:

(1) Observe the height of the barometer at the two stations: observe also the temperature of the mercury by means of a thermometer attached to the barometer, also the temperature of the air in the shade by means of another thermometer detached from the barometer.

(2) Let the observations at the upper and lower station be made as nearly at the same time as may be. And let the observed altitudes of the barometer be reduced to the same temperature, that is, reduced to what they would have been if the temperature at the two stations had been the same; this may be done by augmenting the height of the mercury in the colder temperature, or diminishing that in the warmer temperature, by its $9600^{\text{th}}$ part for every degree of difference between the two; and the altitudes of the mercury so corrected are what are signified by $h$ and $h'$ in the above formula.

(3) Take from a table of logarithms calculated to 7 places of decimals the logarithms of the two heights of the barometer so corrected, and subtract the less from the greater, cutting off from the right-hand side of the remainder three places of decimals; the number to the left of the decimal point will represent the result in fathoms.

(4) This result however must be corrected for the difference of temperature of the air at the two stations as follows: take half the sum of the two temperatures indicated by the detached thermometers as the *mean* temperature; and for every degree by which this differs from the standard tem-

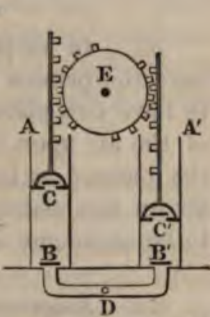
perature of 31°, take so many times the 435th part of the number of fathoms above found, and add them to that number if the mean temperature be more than 31°, subtract them if less.

## THE AIR-PUMP.

In many scientific experiments it is necessary to exhaust the air from vessels made use of. This is done by means of the air-pump*; there are several varieties, some of which effect a more complete exhaustion than others, but none are capable of producing a perfect vacuum. We shall describe two constructions.

37. *Hawksbee's or the common Air-pump.*

$AB$, $A'B'$ are two hollow cylinders, communicating at their lower extremities by a pipe with a strong vessel or receiver, from which it is required to exhaust the air; $B$, $B'$ are valves opening upwards; $C$, $C'$ pistons fitted to rods which are worked by means of a toothed wheel $E$, and containing valves also opening upwards.




Suppose the piston $C$ to be in its highest position, and therefore $C'$ in its lowest, and suppose the density of the air in the receiver to be that of atmospheric air; then when $C$ descends and $C'$ rises, the valve $B$ closes, and $C$ opens because the pressure below becomes greater than that of atmospheric air; also $B'$ opens and $C'$ is closed, and the air which before occupied the receiver now occupies the receiver and the interior of the cylinder $A'B'$, and is therefore rarefied. At each stroke a similar rarefaction takes place; and thus the air in the receiver is gradually exhausted.

* The first air-pump was constructed by Otto Guericke of Magdeburg in 1654; it was a very rude and inconvenient instrument, and is still preserved as a curiosity in the Royal Library at Berlin. Boyle independently and almost simultaneously produced in England a more convenient instrument.

38. *To find the density of the air in the receiver after* n *turns of the wheel.*

Let $A$, $B$ be the capacities of the receiver and of each of the cylinders respectively, $\rho_n$ the density of the air after $n$ turns, $\rho$ the density of atmospheric air. Then after one turn the air which occupied previously the space $A$ occupies the space $A + B$;

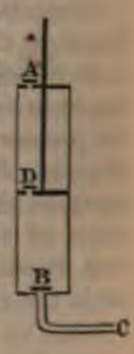
$$\therefore\ \rho_1(A+B) = \rho A, \text{ or } \rho_1 = \rho\frac{A}{A+B};$$

similarly, $\rho_2(A+B) = \rho_1 A$, or $\rho_2 = \rho\dfrac{A^2}{(A+B)^2}$:

and so generally, $\rho_n = \rho\dfrac{A^n}{(A+B)^n}$.

39. In air-pumps, which, as in the above construction, have the pistons open to the atmosphere, it is quite necessary to have two pistons; for if there were only one, the pressure of the air upon it would make it almost impossible to work the pump: by having two, as described, the pressure of the air on the pistons is in equilibrium, and the only resistance to be overcome is that arising from friction.

40. *Smeaton's Air-pump.*




$AB$ is a hollow cylinder communicating with the receiver by a pipe $BC$; $B$ is a valve opening upwards; a piston works in $AB$, having the valve $D$ opening upwards; and the cylinder is closed by a plate, having a valve $A$ also opening upwards.

Suppose the piston in its lowest position; when it rises the valve $B$ opens, $D$ shuts, and the air which occupied the cylinder is expelled through $A$; when the piston descends, $A$ closes, $D$ opens, $B$ closes, and by raising it again the air occupying the cylinder is again expelled, and so on.

41. *To find the density of the air in the receiver after* n *ascents of the piston.*

Let $A$, $B$ be the capacities of the receiver and cylinder respectively; $\rho$ the density of atmospheric air, $\rho_n$ the density after $n$ ascents of the piston. Then after one ascent, the air which occupied the space $A$ occupies the space $A + B$;

$$\therefore\ \rho_1 (A + B) = \rho A, \text{ or } \rho_1 = \rho \frac{A}{A+B},$$

similarly,

$$\rho_2 (A + B) = \rho_1 A, \text{ or } \rho_2 = \rho \frac{A^2}{(A+B)^2},$$

and so generally,

$$\rho_n = \rho \frac{A^n}{(A+B)^n}.$$

42. In this pump only one cylinder is required, the upper surface of the piston not being exposed to the atmospheric pressure. Also the exhaustion producible is much greater than by Hawksbee's construction, because the valve $D$ not being exposed to the air will open for a much longer time than the valves $C$, $C'$ in the former case, which are so exposed.

43. The valves in these pumps are usually formed of a small triangular piece of oil silk, fastened by the corners over an aperture in a brass plate. The receivers are glass vessels of a bell form, which stand upon a brass plate, through which the pipe enters which communicates with the cylinder or cylinders; the junction of the receiver with the brass plate is made air-tight with some greasy substance, or sometimes a disk of leather is interposed. This form of the receiver is necessary for strength, since after a few ascents of the piston the pressure of the atmosphere becomes very considerable: if a cylindrical vessel, open at both ends, be placed upon the plate of the air-pump, and a piece of glass laid horizontally upon the upper end of the cylinder, a few turns of the pump will be sufficient to burst the glass with violence.

## ON THE THERMOMETER.

44. The thermometer is not, properly speaking, a hydrostatical instrument; nevertheless, as we have had frequently

to speak of the *temperature* of fluids, it will be well to describe the instrument by means of which temperature is measured.

The effect of heat is to expand bodies under its influence; this property of bodies is taken advantage of to measure the degree of heat to which they are exposed.

45. The common thermometer consists of a glass tube of small uniform bore, closed at one end and terminating in a bulb at the other, which together with part of the tube is filled with mercury*; the part of the tube not occupied by mercury is a vacuum. The actual filling of the tube is a matter of considerable practical difficulty, but the method of doing it will not be entered upon here. A graduated scale is attached to the tube: when the thermometer is exposed to heat the mercury expands and rises in the tube; the degree of its expansion is known by the graduated scale.

46. The scale is graduated as follows. The thermometer being immersed in melting snow, a mark is made opposite to the surface of the mercury: this is the *freezing point*. The thermometer is next exposed to the steam of water boiling under a given atmospheric pressure, and a mark is made opposite to the surface of the mercury in this case; this is the *boiling point*. The interval between these two points is divided into a number of equal parts called *degrees*: in the centigrade thermometer the freezing point is called $0^0$ and the boiling $100^0$: in Fahrenheit's, the scale commonly used in this country, the former is marked as $32^0$ and the latter $212^0$.

The centigrade graduation is incomparably better than that of Fahrenheit, depending as it does upon a simple intelligible principle, while that of Fahrenheit is based upon an exploded error. Fahrenheit, of Amsterdam, the first who constructed mercurial thermometers, produced a very intense degree of cold by means of a mixture of snow and sea salt, and erroneously imagining this to be the greatest degree of cold possible, he marked it upon his scale as zero; his other limit was the boiling of mercury, and this he marked as $600^0$; and thus the freezing point for water, the most obvious zero point

* Or with coloured spirit of wine.

of graduation, is $32^0$. It is curious that the force of habit should be sufficient to prevent the universal introduction of the centigrade system.

47. *To compare the scales of two differently graduated thermometers.*

Let $C^0$ and $F^0$ denote the number of degrees indicated under the same circumstances by a centigrade and a Fahrenheit's scale. Then $F^0 - 32^0$ is the number of degrees Fahrenheit above the freezing point. Now a degree centigrade measures one hundredth part of the distance from the freezing to the boiling point, and a degree Fahrenheit measures one hundred and eightieth part of the distance;

$$\therefore\ C \times \frac{1}{100} = (F - 32)\frac{1}{180};$$

$$\text{or } C = \frac{5}{9}(F - 32);$$

a formula by means of which we can deduce the reading of one scale from that of the other.

The same method is applicable to the comparison of any two scales.

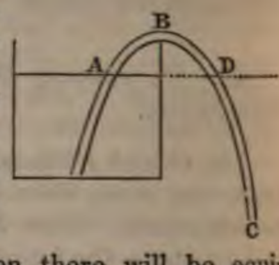
48. We can now reduce any question involving considerations of temperature to numbers; for if we speak of $t$ degrees of temperature, we mean that the mercury in a thermometer exposed to the degree of heat in question would stand at $t$ degrees above the zero point.

For instance, in the case of elastic fluids we have found (Art. 34, Cor.) that $p = k\rho$, provided the temperature is constant; when the temperature varies, the following is the formula given by experiment,

$$p = k\rho(1 + \alpha t),$$

where $t$ is the temperature of the fluid, and $\alpha$ a small quantity, the value of which is found by experiment. The value of $\alpha$ is found to be the same for all gases.

## ON THE SIPHON.




49. The siphon is a bent tube $ABC$ open at both ends. Let the tube be filled with fluid, and the shorter leg inserted into a vessel of fluid which it is required to empty, and the extremity of the other leg closed. Let the level of the surface of the fluid meet the two legs of the siphon in $A$ and $D$ respectively, then there will be equilibrium, provided the height of $B$ above $AD$ is not greater than that of a column of water the weight of which is equal to the atmospheric pressure, and the pressure at $A$ will be equal to that at $D$; consequently the pressure on the end $C$, which we have supposed to be closed, is greater than the atmospheric pressure, and therefore if the tube be opened the fluid will descend: the atmospheric pressure on the surface of the fluid will cause it to rise in the shorter leg, and thus a continuous stream will be produced, which will only cease when the surface of the fluid has descended to the extremity of the shorter leg of the siphon.

The limit of the height of $B$ above the level of the surface of the fluid is about 34 feet.

The principle of the siphon enables us to explain the phenomenon of intermitting springs. Suppose a siphon to be introduced into the side of a vessel, the highest point $B$ of the siphon not being the highest point of the vessel but any point in its side; and suppose that we pour water into the vessel; it will rise within the shorter leg of the siphon as well as in the vessel round about it, until the level of the water is higher than $B$, it will then begin to run down through the longer leg of the siphon, and will continue to do so until the vessel is emptied. If then we cause water to percolate slowly into a vessel furnished with a siphon as above described, we shall have this result, that there will be an intermitting flow of water from the longer leg of the siphon; for the vessel will continue to fill until the level of the water is above the highest point of the siphon, the vessel will then be emptied

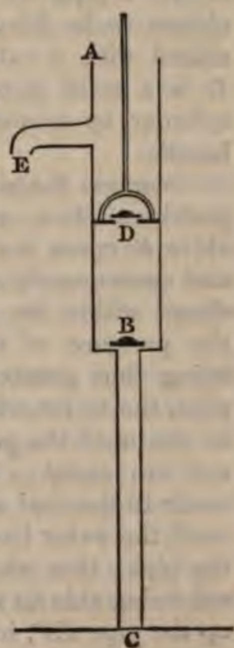
by the siphon, then it will begin to fill again, and the process will be repeated. Now the conditions here assigned may be easily fulfilled in the case of an internal cavity in a rock or hill, out of which there may be a tortuous fissure forming a natural siphon, and if there be other fissures through which the cavity can gradually fill with water, and if lastly the fissures supplying the reservoir have free communication with the atmospheric air, all the conditions necessary for an intermitting stream of water from the opening of the siphon-shaped fissure will be satisfied.

## ON PUMPS.

50. The pressure of the atmosphere on the surface of water is taken advantage of, for the purpose of raising it above its level. The machine by means of which this is effected, is called a *pump;* we shall describe two principal kinds.

### *The Common Pump.*

51. $AB$ is a cylinder, having its lower end closed with a valve $B$ opening upward, and connected by means of a pipe $BC$ with the water which is to be raised. A piston, containing a valve $D$ opening upwards, is worked in the cylinder by means of a vertical rod and a handle.




Suppose the piston to be in its lowest position; then, when it is raised, the valve $B$ opens and a partial vacuum is produced in the cylinder and pipe, and the pressure of the atmosphere without being greater than the pressure within the pipe, the water rises, and it continues to rise until the pressure within and without become equal. When the piston descends, the valve $D$ opens, and the air within the cylinder escapes; when it is raised, the former process is repeated, and

so on until the water rises to the level of the pipe $E$, from which it escapes.

It is obvious that the length of $BC$ must not be greater than the height of a column of water, the weight of which is equal to the atmospheric pressure, that is, than about 34 feet.

52. The common pump is limited in respect of the height to which it can raise water; but we can raise water to any height by means of the forcing pump; it is by this means that cisterns at the higher part of houses are supplied.

*The Forcing Pump.*

53. $AB$ is a cylinder, having at its lower end a valve $B$ opening upwards, and connected by a pipe $BC$ with the water to be raised. From the lower part of $AB$ a pipe $EF$ communicates with the cistern to be filled, and this pipe is furnished with a valve $E$ opening upwards. $D$ is a solid piston which works in the cylinder by means of a vertical rod and handle.

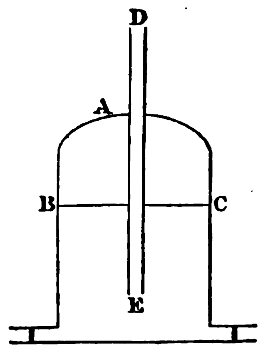

Suppose the piston to be in its lowest position; then, when it is raised, the valve $B$ opens and the valve $E$ is closed, and consequently a partial vacuum is produced within the cylinder and pipe; and the pressure of the atmosphere without being thus greater than that within the pipe, the water within rises, and continues to rise until the pressures within and without are equal. Let the piston be now made to descend and the process repeated, until the water has risen above the top of the pipe; then when $D$ descends, the water in the cylinder, not being able to return on account of the valve $B$, is forced up the pipe $EF$, in which it is retained by the valve $E$.

This process may be continued, and by this means water may be raised to any elevation.

As in the case of the common pump, the height of the valve $B$ above the level of the water must not exceed 34 feet.

*The Fire-engine.*

54. The fire-engine consists of two forcing pumps, by means of which the water is forced into an air-vessel $ABC$, from which the water can escape by the pipe $ED$. The air in the vessel being compressed by the water, which is forced in by the pumps, exerts a continuous pressure on the surface of the water $BC$, by which it is driven violently and in a continuous stream through the pipe $ED$. A flexible tube is attached to the mouth of the pipe $ED$, by means of which the stream of water can be made to play in any direction.




## ON THE DIVING-BELL.

55. The diving-bell is a heavy chest, which is suspended by a rope, and which has its lower side open. If the bell be lowered into the water, the air within the bell will prevent the water from filling it, and consequently persons sitting on a seat inside will be enabled to breathe at considerable depths below the surface of the water.

In practice the diving-bell is furnished with a flexible pipe communicating through the top of the bell with the interior, by means of which fresh air can be pumped in, and the interior thus kept as free from water as we please, while at the same time fresh air is furnished for the respiration of the divers.

Let $B$ be the volume of air contained by the bell, $B_z$ the volume it contains when at the depth $z$ below the surface,

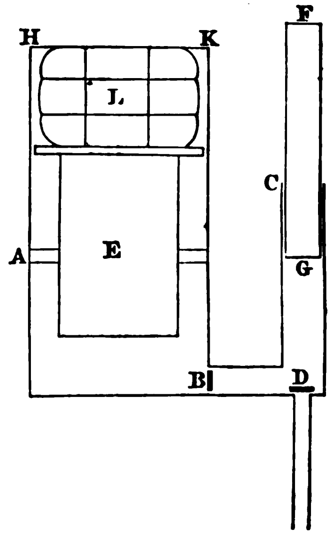
$h$ the height of a column of water the weight of which equals the atmospheric pressure, then (by Arts. 10 and 34)

$$\frac{B_z}{B} = \frac{h}{h + z}.$$

The diving-bell is extensively applied in submarine operations; the raising of the wreck of the Royal George may be quoted as an instance. It was first described by Dr Halley, who explained its defects and suggested remedies for them. The greatest inconvenience in its application arises from the painful pressure upon the ears of the diver; it would seem that there are cavities in the ear opening outwards, and that by pores so small, as not to give admission to the air itself, unless they be distended by a considerable force. Hence on the first descent of the bell, a pressure is felt which soon becomes painful, until it is relieved by the sensation of something bursting within the ear; as the bell descends lower the pain is renewed, and relieved in like manner. In Dr Halley's experiment a diver attempted to avoid this unpleasant effect by putting paper into his ears; the result was that the paper was driven into the ears in such a manner as to be with difficulty extracted by medical aid.

## ON BRAMAH'S PRESS.

56. The principle of the equal transmission of fluid pressure in all directions, and the consequent possibility of increasing the total pressure on a surface to any extent by increasing the surface, supply us with the means of obtaining one of the most powerful machines in use, the application of which to the purpose of producing enormous pressure or tension is extremely valuable. One of its most interesting applications has been to the raising of the tubular bridge across the Menai Straits.




A solid cylinder $E$ works through a water-tight collar in the end of the strong hollow cylinder $AB$; the latter is connected by a pipe, having a valve $B$ opening inwards, with another strong cylinder $CD$, which with the solid cylinder $FG$ acting as a piston forms a forcing pump. Suppose the machine to be used for compressing a bale of goods $L$; then the bale is placed upon $E$, and is pressed by it against the very strong framework $HK$.

When the pump is worked the water is forced through $B$, which pressing on the lower surface of $E$ causes it to rise, and to compress $L$; and the pressure may be increased by continuing to work the pump, the force with which the piston $FG$ descends at each stroke being multiplied in its effect upon $L$ by the ratio of the area of the base of the piston $E$ to that of the piston $FG$.

The pressure may be immediately relieved, by allowing the water in $AB$ to escape by a cock provided for the purpose.

Let $r$ and $R$ be the radii of the cylinders $FG$ and $E$ respectively, $p$ the pressure referred to a unit of surface in the fluid within the press, $P$ the force applied at the extremity of the pump-handle, $A$ and $a$ the distances of the fulcrum of the pump-handle from the extremity at which $P$ acts and the other extremity respectively, $W$ the pressure on the cylinder $E$; then

$$W = p\pi R^2,$$

$$\text{and } PA = p\pi r^2 a,$$

$$\therefore \frac{P}{W} = \frac{r^2 a}{R^2 A}.$$

This formula gives us the measure of the mechanical advantage of the machine; and we may observe that the result is in accordance with the principle of Virtual Velocities.

## ON THE STEAM-ENGINE*.

57. The account which we shall here give of the steam-engine will be exceedingly brief; we shall in fact consider it principally as a hydrostatical machine, whereas the complete view of it would represent it as involving the principles of dynamics, and would require a description of a variety of ingenious contrivances which would here be out of place. It has only been by degrees that the steam-engine has attained to its present perfection; we shall describe it as it was completed by James Watt, to whose genius the most important of the contrivances are due.


$C$ is a piston working in a cylinder $AB$, the rod $CD$ which communicates the motion of the piston to the machinery passing through a steam-tight collar at $A$. At $E$ and $F$ two pipes, which communicate with the boiler, enter the cylinder; suppose one of these to be open to the boiler and not the other, (in the figure the lower pipe is open to the boiler, the upper not,) then the steam rushing in through $F$ below the piston will drive it up; when the piston is at the highest point of its stroke, suppose this arrangement reversed, that is, $E$ to be opened to the boiler and $F$ not; then the steam rushing in above the piston will

* It is quite impossible to give an adequate view of the steam-engine within the limits necessarily prescribed to this article: the student is recommended to consult a work treating of the Steam-engine and its applications in detail; probably the *Rudimentary Treatise* in Weale's series will be found sufficient.

drive it downward, and the steam which is below the piston will escape through $F$ either into the outer air or into a vessel provided for the purpose, it being so contrived that the arrangement which opens $E$ to the boiler cuts off the communication of $F$. If this alternate opening of the pipes be effected by some contrivance, it is manifest that we shall obtain a continuous oscillating motion of the piston $C$.

The contrivances for the alternate opening of the pipes are various; in the figure we have represented a simple and common one. $G$ is a steam-tight box which is made to slide up and down by means of the rod $GH$; the magnitude of this box or slide is such that when one pipe is just covered the other is just uncovered, and while one of the steam-pipes, as $F$ in the figure, is uncovered by it, the other ($E$) communicates through it with the vent-pipe $K$. If then the rod $HG$ has exactly the reverse motion of the rod $CD$, so that one shall rise when the other falls, it is evident that what was required will be done.

The oscillatory motion of the rod $CD$ is converted into a rotatory motion as follows. The extremity $D$ is connected by means of a system of rods with one end of a beam, which works in a vertical plane about a horizontal axis through its middle point. This system of rods is known under the name of *parallel motion*, and was devised by Watt; the ingenuity of the contrivance consists in the manner in which a connection is effected between the point $D$ which moves in a straight line, and the extremity of the beam which describes an arc of a circle. The other extremity of the beam is made by means of a crank to turn a heavy wheel, called the *flywheel;* this wheel performs a very important function, for its momentum is such as to carry the machinery past certain positions, known as the *dead points*, and also to equalize the motion throughout. The use of such a wheel is not peculiar to the steam-engine, but is applied to many common machines, as for instance the knife-grinder's wheel and the turning lathe.

The motion of the rod $HG$ is produced by an excentric crank upon the axis of the flywheel.

The steam, when allowed to escape from the cylinder, through the pipe $K$, may be permitted either to escape into

the outer air, or else to flow into a closed vessel in which it is condensed, and the water formed by it pumped up again into the boiler to be reconverted into steam.

In the former case the pressure of the atmosphere is allowed to act on the piston, and must be overcome by the greater pressure of the steam on the opposite side: hence, in order to produce a given effect, the elastic force of the steam, and consequently the quantity of fuel employed, must be much greater in this case, than when the steam is allowed to escape into a condenser; engines on the former construction are called *High Pressure*, on the latter *Low Pressure* engines.

The high pressure engine has also this disadvantage, that the interior of the cylinder is cooled by being open to the atmosphere, and consequently when the steam is again admitted some portion is condensed and rendered ineffective.

It is usual to cut off the steam before the piston has attained to the end of its stroke, and to allow the stroke to be completed by the elastic force of the steam already injected; by this arrangement not only is less steam required, but also the motion is more uniform.

58. An earlier and imperfect form of the steam-engine was the *atmospheric* engine, in which the piston was driven up by steam, and a vacuum having been produced below it by injection of cold water the piston was driven down by the atmospheric pressure above. The capital defect of this construction is, that the cylinder must be cooled down at every stroke, and consequently when the steam is again admitted a very large quantity is condensed, and there is an immense waste of fuel. This construction has consequently almost entirely disappeared.

# OPTICS.

# OPTICS.

1. The science of which we are now about to explain the principles is known as that of *common* or *geometrical* optics, in contradistinction from *physical* optics; in this latter science, we endeavour by means of a simple hypothesis concerning the constitution of light, to connect and account for the various phenomena presented to us; in the former, we are principally employed in tracing, by mathematical calculation, the results of certain experimental laws. Hence, the conclusions arrived at in the following pages, will be equally sound, whatever physical theory be adopted, and the subject will be, for the most part, one of pure geometry.

2. From a bright object light emanates in all directions, and this light we may conceive to be made up of *rays*, intending by the term *ray* to express the smallest quantity of light which can proceed in any direction; and we reason concerning rays, as though they were geometrical lines. In like manner, an object to be a source of light must be of finite, though it may be of very small dimensions; but we shall consider a bright point which is a source of light as a geometrical point.

3. Any substance which allows the transmission of light through it is called a medium. Light may proceed either through a medium or in vacuum.

4. An assemblage of rays proceeding from a luminous point is called a *pencil* of rays. The pencils which we shall consider will be conical, and the axis of the cone will be called the axis of the pencil.

A conical pencil may consist either of divergent or convergent rays: if the rays are proceeding from a luminous point, the pencil is divergent; if the rays are proceeding from some source of light towards a point, it is convergent; if the

rays are parallel, the pencil is neither divergent nor convergent.

5. When a ray of light is proceeding in a uniform medium or in vacuum, its direction is rectilinear, but when it is incident upon the surface of a medium, it is in general divided into three parts,

(1) One portion is reflected according to a regular law, and forms the *reflected ray;*

(2) Another portion enters the medium according to a regular law, and forms the *transmitted* or *refracted ray;*

(3) A third part is *scattered,* that is, reflected in all directions without any regular law.

The first two portions mentioned are those with which we shall be hereafter concerned, the third part is that which renders the surfaces of bodies ordinarily visible.

Suppose, for example, we are looking at the surface of a well polished mirror, then the effect of the mirror is only to *reflect* rays of light which fall upon it from various objects, and we shall not be sensible of the existence of the mirror itself, but if there be any speck upon the surface of the mirror this speck will *scatter* the light which falls upon it, and become visible exactly as if it were itself a luminous object. In fact when rays of light are reflected, or refracted, their directions only are changed; but when a pencil of light is scattered, the object which scatters it becomes virtually a new source of light.

Besides the reflected, refracted, and scattered light, there is also a certain portion *absorbed* by the medium.

In the case of polished metallic surfaces and some others, the reflected ray is the only one which sensibly exists; and, in general, the relative intensities of the reflected and refracted rays, will vary with the circumstances of the incidence, and also with the nature of the medium.

6. When a ray of light is incident upon a plane surface, the angle which its direction makes with the line perpendicular to the surface, or the *normal* to the surface, is called the *angle of incidence,* and the angles which the reflected and

refracted ray respectively make with the same line are called the *angles of reflexion and refraction.* When a ray is incident on a curve surface, the ray will be reflected or refracted in the same manner as if it fell upon the plane which touches the surface at the point of incidence, and the angles of incidence, reflexion, and refraction are those which the incident, reflected, and refracted ray respectively make with the normal to this plane.

7. The laws of reflexion are the following:

(1) *The incident and reflected ray lie in the same plane with the normal at the point of incidence, and on opposite sides of it.*

(2) *The angles of incidence and reflexion are equal.*

And the following are the laws of refraction:

(1) *The incident and refracted ray lie in the same plane with the normal at the point of incidence, and on opposite sides of it.*

(2) *The sine of the angle of incidence bears to the sine of the angle of refraction a ratio dependent only on the nature of the media between which the refraction takes place, and on the nature of the light*.*

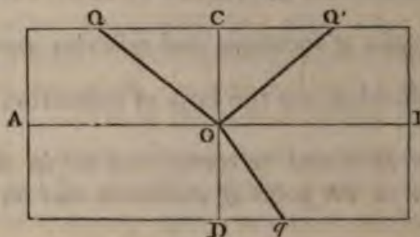
According to this last law, if we call the angle of incidence $\phi$, and that of refraction $\phi'$, we shall have $\sin\phi = \mu \sin\phi'$, where $\mu$ is a quantity independent of the angle of incidence, and depending only upon the nature of the media and of the light; it will have for instance a certain value for refraction from vacuum into glass, another from glass into water, and so on; also it will have one value for red light, another for

* The true Law of Refraction was first discovered by Willebrord Snell, professor of Mathematics at Leyden; who found by experiment, that the cosecants of the angles of incidence and refraction are always in the same ratio. The discovery was at first erroneously attributed to Des Cartes. Snell was born at Leyden in 1591, became professor of Mathematics in 1613, and died in 1626.

green, and so on. The quantity $\mu$ is called the refractive index, and is greater than 1 when refraction takes place from vacuum into a medium, and in general is greater than 1 when the refraction is from a rarer to a denser medium, and less than 1 when the opposite is the case.

8. These laws may be considered as depending for their truth upon experiment; in a treatise on Physical Optics they would be deductions from an hypothesis respecting the constitution of light, but in a treatise like the present they may be regarded as experimental truths.

The following experiment will serve to deduce the laws from actual observation, and with proper precautions is susceptible of considerable accuracy.




Take a rectangular card, the opposite sides of which are bisected by the lines $AOB$, $COD$, immerse it perpendicularly in water as far as the line $AB$, and place it in such a position that a very small beam of sunlight, admitted through an orifice in a shutter of a darkened room, may be incident along the line $QO$ on the surface of the water at $O$.

Then a portion of this ray will be observed to be reflected in such a direction as $OQ'$, and on measuring $CQ$, $CQ'$, they will be found to be equal: hence it will be seen that $QOC = Q'OC$; and it is manifest that $QO$, $CO$, $Q'O$ are in the same plane, they being all in the plane of the card.

Again, a ray $Oq$ will be observed to be transmitted through the water; this is the refracted ray, and is manifestly in the same plane with $QO$ and $OD$; also if for different angles of incidence the lines $CQ$, $Dq$ be measured, and $OQ$, $Oq$ computed from them, it will be found that the ratio $\frac{CQ}{OQ} : \frac{Dq}{Oq}$ is

the same, whatever be the direction of the ray. The ratio however will not be the same, if another fluid be substituted for water, or if the colour of the light be varied.

The results here described are in accordance with the laws enunciated in the preceding article; and it is evident that experiments may be made, upon principles similar to that of the one just now described, sufficiently numerous and accurate to establish the laws with a very great degree of certainty. But besides the evidence thus arising from direct observation, we can appeal to the coincidence with fact of the results of calculations founded upon these laws; in the case of reflexion, for instance, a method of observing the heavenly bodies depends for its accuracy upon the law of reflexion which has been enunciated, and the slightest deviation from the truth in this assumed law would be certain to be detected in a long course of observations*. So that on the whole we are entitled to look upon the laws which we have enunciated as accurate physical laws.

9. Prop. *If the refractive index for a medium* (A) *when light is incident upon it from vacuum be* $\mu$, *and the index for another medium* (B) *under the same circumstances be* $\mu'$, *then when light proceeds from* (B) *into* (A) *the refractive index is* $\frac{\mu}{\mu'}$.

The proof of this proposition depends upon the two following experimental laws:

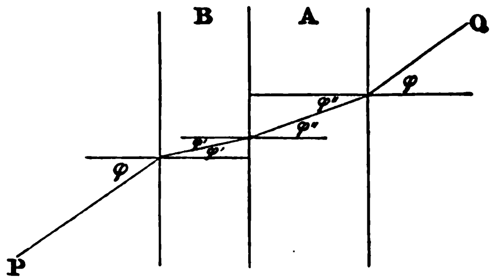
(1) If a ray of light proceed from a point $P$ to another $Q$, suffering any reflexions or refractions in its course, then if it be incident in the reverse direction from $Q$ it will follow the exactly reverse course to $P$.

(2) If a ray pass from vacuum, through any number of media having their surfaces plane and parallel, when the ray emerges into vacuum its direction will be parallel to that which it had before incidence.

Now let $\phi$ be the angle of incidence from vacuum upon the medium $B$, $\phi'$ the angle of refraction, which will also be

* See *Astronomy*. Art 23.

the angle of incidence upon the medium $A$. Also let $\phi''$ be the angle of refraction into $A$, which will also be the angle of incidence upon the second bounding surface of $A$; and by the second of the preceding laws the angle of emergence into vacuum will be $\phi$. Hence we shall have




$$\sin\phi = \mu' \sin\phi',$$

and by the first of the above laws,

$$\sin\phi = \mu \sin\phi'';$$

$$\therefore \sin\phi' = \frac{\mu}{\mu'} \sin\phi'';$$

which proves the proposition.

COR. If the refractive index from vacuum into a medium be $\mu$, that from the medium into vacuum will be $\frac{1}{\mu}$.

## ON THE CRITICAL ANGLE.

10. Let $\phi$ be the angle of incidence of a ray within a medium, the refractive index of which is $\mu$, and $\phi'$ the angle of refraction into vacuum; then

$$\sin\phi = \frac{1}{\mu} \sin\phi'.$$

From this formula if $\phi$ be given $\phi'$ may be found, and a real value will be given so long as $\sin\phi$ is less than $\frac{1}{\mu}$; but when $\phi$ has a value greater than that determined by the equa-

tion $\sin \phi = \frac{1}{\mu}$, the formula fails to give us a value of $\phi'$, it becomes in fact impossible, because the sine of an angle cannot be greater than unity. In consequence of this failure of our formula we have recourse to experiment, and we find that in reality there is no refracted ray when the angle of incidence is greater than that above assigned, the ray being wholly reflected within the medium. The angle of which the sine is $\frac{1}{\mu}$ is called the *critical angle*. The critical angle for glass is about $41^\circ 45'$, for water about $48^\circ 30'$.

This internal reflexion at the surfaces of media is the most complete kind of reflexion, that is to say, the reflected light is more nearly equal in intensity to the incident than in any other case. The critical angle is sometimes called the angle of *total reflexion*. Refraction from vacuum into a medium, or from a rarer into a denser medium, is always possible.

The effect of total reflexion may be easily exhibited by filling a common drinking glass with water, and holding it above the level of the eye; if we then look upwards through the water, we shall see the whole surface shining as with a strong metallic reflexion. The most remarkable result however is that which takes place in the case of vision by an eye under water. An eye so situated will see all external objects through a circular aperture of about $97^\circ$ in diameter overhead; all objects down to the horizon will be seen in this space, but those near to the horizon much distorted and contracted in dimensions. Beyond the limits of this circle will be seen the bottom of the water and all objects in the water, by total reflexion from the surface.

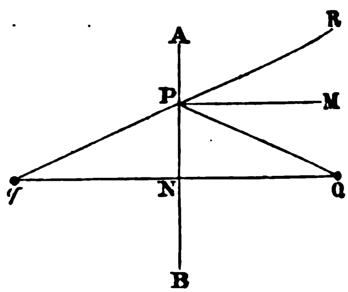
11. We shall now proceed to investigate the effect produced upon the form of a small pencil of rays, when reflected and refracted under various conditions: the breadth of the pencil will, in general, be considered indefinitely small, for the sake of mathematical convenience, and our results must therefore be regarded as an approximation to the actual case, in which the breadth of pencils, though generally very small, is

of course not indefinitely small; for many purposes the results we shall obtain will be as useful, as if the approximation had not been made. We shall first consider some cases of reflexion, and then some of refraction.

## ON REFLEXION AT A SINGLE SURFACE.

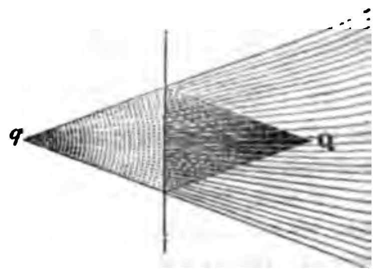
### I. *A Plane Surface.*

12. *A conical pencil of rays is incident upon a plane reflecting surface; to determine the form of the reflected pencil.*




Let $AB$ be a section of the surface made by a plane perpendicular to it, and passing through the luminous point $Q$, or focus of incidence. Draw $QN$ perpendicular to the surface, produce it, and take $Nq = NQ$. Let $QP$ be any incident ray, join $qP$ and produce it to $R$; $PR$ will be the reflected ray.

Draw $PM$ perpendicular to the surface. Then in the triangles $PQN$, $PqN$, we have $NQ = Nq$, $PN$ common, and the angles $QNP$, $qNP$ equal, being right angles; hence the angle $QPN = qPN = APR$: therefore also angle $QPM = RPM$, or $PR$ is the reflected ray.




Hence the ray $QP$ proceeds after reflexion at $P$, as if it came from $q$; and the same may be said of each other ray, therefore all the rays after reflexion proceed as if they came from $q$, and if the incident pencil be a cone having $Q$ for its vertex, the reflected will also be a cone having $q$ for its vertex. An

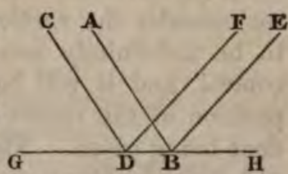
incident and reflected conical pencil are represented in the accompanying figure.

We may call the point $q$ the focus of reflexion; but it is to be observed that it is a *virtual* not a *real* focus, that is to say, the reflected rays proceed not actually, but only *as if* they came, from it. So also the line $qP$, which is the direction of the reflected ray $PR$, may be called a *virtual ray*.

Since the rays after reflexion proceed from $q$ as before reflexion from $Q$, $q$ is sometimes called the *image* of $Q$.

13. *Parallel rays, reflected at a plane surface, continue parallel.*

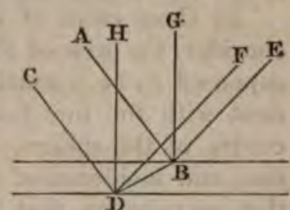
(1) Let the angles of incidence be in the same plane.




Let $GH$ be the reflecting surface; $AB$, $CD$ two incident rays, $BE$, $DF$ the corresponding reflected rays.

Then $ABG = EBH$ and $CDG = FDH$, by the law of reflexion. And $ABG = CDG$, since the incident rays are parallel; therefore $EBH = FDH$, or the reflected rays are parallel.

(2) Let the angles of incidence be in different planes.




Let $AB$, $CD$ be two incident rays; $BG$, $DH$ perpendiculars to the reflecting surface at $B$ and $D$; join $BD$; and let $BE$ be the reflected ray corresponding to $AB$. Also let $DF$ be the intersection of the planes $EBD$, $CDH$.

Then $BG$, $DH$ being perpendicular to the same plane are parallel; (Euc. XI. 6) and $AB$ is parallel to $CD$ by hypothesis; therefore the angles $ABG$, $CDH$ are equal, and therefore the angles of reflexion are equal.

Again, since $AB$, $BG$ are respectively parallel to $CD$, $DH$, the planes $ABG$, $CDH$ are parallel, (Euc. XI. 15); and they are intersected by the plane $EBDF$, therefore $BE$ is parallel to $DF$; therefore the angles $GBE$, $HDF$ are equal; but

$GBE$ is the angle of reflexion for the ray $AB$, therefore $HDF$ is equal to the angle of reflexion for the ray $CD$. And $HDF$ is in the same plane with the angle of incidence $CDH$; therefore the reflected ray corresponding to $CD$ is $DF$, which has before been shewn to be parallel to $BE$.

## II. *A Spherical Surface.*

14. We have seen that however large a cone of rays is incident upon a plane surface, the reflected pencil is accurately conical; but this, it is easy to see, will not be the case when rays are incident on a spherical surface. If, however, we consider the vertical angle of the incident conical pencil to be indefinitely small, the reflected pencil will also be conical, and it will be our business now to investigate the position of the vertex of the reflected cone, that of the incident being given. This vertex is called the *geometrical focus*, and if the incident rays are parallel, it is called the *principal focus of the mirror*. Also if $Q$ be the focus of incident rays, $q$ the geometrical focus of the reflected rays, it is manifest from the first of the two general laws quoted in Art. 9, that if $q$ be made the focus of incident rays, $Q$ will be the focus of reflected rays; hence $Q$ and $q$ are called with reference to each other *conjugate foci*.

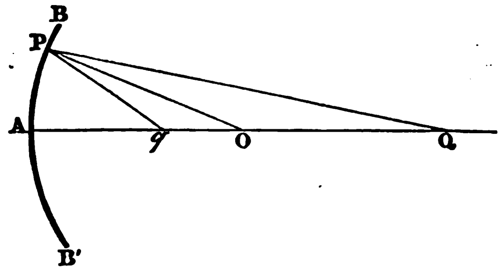
In those cases of incidence which we are now about to consider, the axis of the small incident conical pencil will be supposed to be normal to the surface, and therefore coincident with the line joining the point of incidence and the centre of the sphere. This line is called the *axis of the surface*, and incidence of this kind is called *direct* incidence; so that we may say, that incidence is direct when the axis of the incident pencil and that of the surface coincide. Or we may say, that incidence is direct when the axis of the incident pencil passes through the centre of the sphere.

If the axis of the incident pencil is inclined to that of the reflector, the incidence is said to be *oblique*.

The calculations which follow will all refer to the case of direct incidence.

15. *Diverging rays are incident upon a concave spherical reflector; to find the geometrical focus.*

Let $BAB'$ be a section of the reflector, made by a plane passing through the centre $O$ of the sphere, and the focus of incidence $Q$. Let $QP$ be any incident ray, $QOA$ the axis of the conical pencil: join $OP$, and make $OPq = OPQ$, then




$Pq$ is the reflected ray; and the ultimate position of $q$, when $P$ moves up to $A$, will be the geometrical focus.

We have, by Euclid, VI. 3, since $QPq$ is bisected by $PO$,

$$\frac{Pq}{qO} = \frac{PQ}{QO},$$

but ultimately, $Pq = Aq$, and $PQ = AQ$;

$$\therefore \frac{Aq}{qO} = \frac{AQ}{QO}.$$

If we denote $AQ$ by $u$, $Aq$ by $v$, and $AO$ by $r$, we have

$$\frac{v}{r-v} = \frac{u}{u-r},$$

$$\text{or } \frac{r}{v} - 1 = 1 - \frac{r}{u},$$

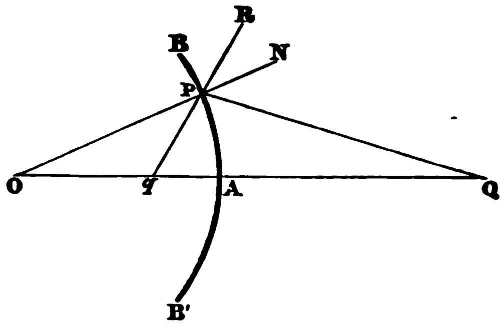
$$\text{or } \frac{1}{v} + \frac{1}{u} = \frac{2}{r}.$$

This equation determines $v$, when $u$ is given.

16. *The same proposition for a convex surface.*

In this case we must draw $PR$, to make with $OP$ produced to $N$ the angle $RPN = QPN$, and produce $RP$ to cut the axis in $q$.

Then since the external angle $RPQ$ of the triangle $QPq$ is bisected by $OPN$, therefore by Euclid, VI. A,




$$\frac{Pq}{qO} = \frac{PQ}{QO};$$

$$\therefore \frac{Aq}{qO} = \frac{AQ}{QO};$$

and using the same notation as before, we shall have

$$\frac{v}{r - v} = \frac{u}{u + r},$$

$$\text{or } \frac{r}{v} - 1 = 1 + \frac{r}{u},$$

$$\text{or } \frac{1}{v} - \frac{1}{u} = \frac{2}{r}.$$

In this case the focus $q$ is *virtual*.

17. In like manner we might investigate the cases of incidence of converging rays. We shall find, however, that all four cases may be brought under one formula, by adopting a convention respecting the sign $-$, as indicating direction, similar to that which we have already found so convenient in other subjects.

It will be found, that, if we suppose light to proceed from right to left across the paper, so that $Q$ is on the right of the mirror for diverging rays, and on the left for converging, the following formulæ will result:

(1) Concave mirror, diverging rays:

$$\frac{1}{v}+\frac{1}{u}=\frac{2}{r}.$$

(2) Concave mirror, converging rays:

$$\frac{1}{v}-\frac{1}{u}=\frac{2}{r}.$$

(3) Convex mirror, diverging rays:

$$-\frac{1}{v}+\frac{1}{u}=-\frac{2}{r}.$$

(4) Convex mirror, converging rays:

$$-\frac{1}{v}-\frac{1}{u}=-\frac{2}{r}.$$

Now let us adopt this convention, that lines shall be positive or negative, according as they are measured towards the source of light, or in the opposite direction; so that $u$ will be positive or negative, according as the incident rays are divergent or convergent; $v$ will be positive or negative, according as the reflected rays are convergent or divergent; and $r$ will be positive or negative, according as the mirror is concave or convex. Then it will be seen that the four preceding formulæ will all be embraced in the following,

$$\frac{1}{v}+\frac{1}{u}=\frac{2}{r}.$$

18. By making $u=\infty$, in the formula

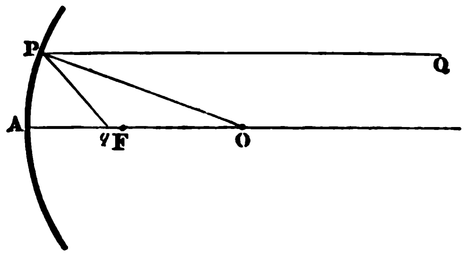
$$\frac{1}{v}+\frac{1}{u}=\frac{2}{r},$$

we find that $v=\frac{r}{2}$, which shews that the principal focus of a mirror is half-way between the mirror and the centre of the spherical surface of which it is formed. This, however, is a sufficiently important proposition to deserve a separate investigation.

*To find the principal focus of a concave mirror.*

Let $QP$ be a ray of light parallel to $AO$, the axis of

the mirror. Join $PO$, $O$ being the centre of the sphere, and make $qPO = OPQ$, then $Pq$ is the reflected ray correspond-



ing to $QP$, and $F$, the ultimate position of $q$ when $P$ moves up to $A$, is the principal focus.

Then the angle $qPO = OPQ = qOP$, since $Oq$, $PQ$ are parallel;

$$\therefore Pq = qO;$$

and this being always true, $AF = FO$;

$$\therefore AF = \frac{AO}{2}.$$

A similar demonstration is applicable to the case of a convex mirror.

$AF$ is called the *focal length* of the mirror, and is frequently denoted by the letter $f$; so that $f = \frac{r}{2}$.

19. The formula $\frac{1}{v} + \frac{1}{u} = \frac{2}{r}$ may be put in a different form, by measuring the distances of $Q$ and $q$ from the point $F$. For we have

$$QF = u - \frac{r}{2}, \qquad qF = v - \frac{r}{2};$$

$$\therefore u = QF + \frac{r}{2}, \text{ and } v = qF + \frac{r}{2},$$

and hence the formula $\frac{1}{v} + \frac{1}{u} = \frac{2}{r}$ becomes

$$\frac{1}{qF + \frac{r}{2}} + \frac{1}{QF + \frac{r}{2}} = \frac{2}{r},$$

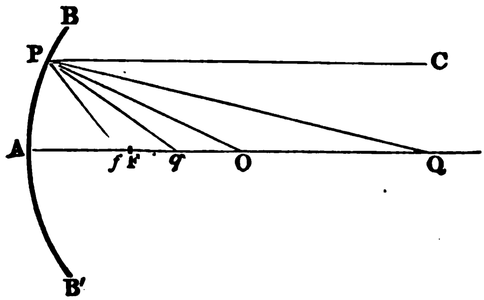
$$\text{or } \frac{r}{2}(QF + qF + r) = \left(qF + \frac{r}{2}\right)\left(QF + \frac{r}{2}\right)$$

$$= qF \, . \, QF + \frac{r}{2}\left\{QF + qF + \frac{r}{2}\right\};$$

$$\therefore qF \, . \, QF = \frac{r^2}{4} = OF^2.$$

[The proposition expressed by the preceding formula we shall enunciate in words, and demonstrate as follows, without reference to the formulæ already established.

20. *When a small pencil of diverging or converging rays is incident directly upon a spherical reflector, the distance of the centre of the sphere from the principal focus is a mean proportional between the distances of the conjugate foci from the same point.*




Let $BAB'$ be the spherical reflector, $O$ its centre, $Q$ the focus of incidence, $QOA$ the axis of the pencil, $QP$ any incident ray; join $OP$, and make $OPq = OPQ$, then the ultimate position of $q$ when $P$ approaches indefinitely near to $A$ will be the geometrical focus.

Draw $PC$ parallel to $AQ$, and make $OPf = OPC$; and bisect $AO$ in $F$.

Then in the triangles $QPf$, $qPf$, we have the angle $f$ common. Also the angle $PQf = CPQ = CPO - OPQ$

$$= fPO - OPq = fPq.$$

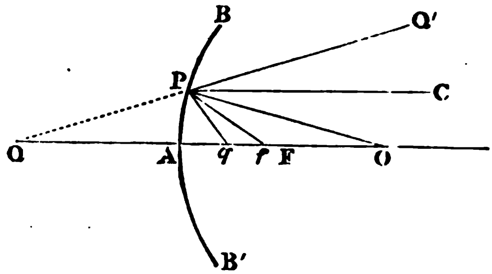
Therefore the remaining angle $QPf$ = angle $fqP$, and the triangles are similar,

$$\therefore Qf : Pf :: Pf : qf;$$

and this is true ultimately when $P$ coincides with $A$, and $f$ with $F$,

$$\therefore QF : AF :: AF : qF.$$

The preceding figure is constructed for the case of divergent rays incident on a concave surface; the same demonstration is applicable, *mutatis mutandis*, when the incident rays converge, as is represented in the annexed figure.




If the lines $CP$, $QP$, $OP$, $qP$, $fP$ be produced, the figures will serve for those cases in which the rays are incident upon the convex surface.]

21. If in the formula $\frac{1}{v} + \frac{1}{u} = \frac{2}{r}$, we give $u$ any value, we obtain the corresponding value of $v$, which, if we attend to its algebraical sign, will make us acquainted with the form of the reflected pencil. Suppose, for instance, $r = 4$ in., $u = \frac{3}{2}$ in., then $\frac{1}{v} = \frac{1}{2} - \frac{2}{3} = -\frac{1}{6}$, or $v = -6$; hence the rays, which diverge upon the mirror from a point one inch and a half to the right, diverge after reflexion from a point six inches to the left of the mirror. And more generally, if we suppose $Q$ to assume all possible positions, we shall be able to find the corresponding positions of $q$; this we proceed to do in the following proposition.

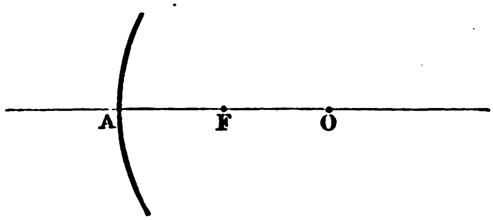
22. *To trace the corresponding positions of the conjugate foci.*

Suppose the mirror to be concave, then we have

$$\frac{1}{v} + \frac{1}{u} = \frac{2}{r}.$$

Since the sum of $\frac{1}{v}$ and $\frac{1}{u}$ is constant, as one of them increases the other must decrease, and the same will be true of their reciprocals $v$ and $u$; hence $Q$ and $q$ always move in opposite directions.

(1) Let $Q$ be at an infinite distance to the right of the mirror, or the incident rays parallel; then $v = \frac{r}{2}$ and $q$ is at $F$.




(2) Let $Q$ move towards $O$; then $q$ moves to meet it; and at $O$ they coincide, because when $u = r$, $v = r$.

(3) Let $Q$ move from $O$ towards $F$; then $q$ moves to the right of $O$; and when $Q$ has reached $F$, $q$ is at an infinite distance, or the reflected rays are parallel, because when $u = \frac{r}{2}$, $v = \infty$.

(4) Let $q$ move from $F$ towards $A$; then $q$ moves from an infinite distance on the left of $A$ to meet it; and when $Q$ has reached $A$, $q$ is there also, because when $u = 0$, $v = 0$.

(5) Let $Q$ move to the left of $A$; then $q$ moves to the right; and when $Q$ has attained to an infinite distance, $q$ is at $F$, because when $u = \infty$, $v = \frac{r}{2}$.

$Q$ and $q$ are now in the same positions as at first, and therefore we have traced all their corresponding positions.

The corresponding positions of the foci for a convex surface may be traced in a similar manner.

Cor. It appears from the preceding investigation, that $Q$ and $q$ are always on the same side of $F$.

23. The formulæ which have been proved for a spherical mirror may be adapted to the case of a mirror formed by the revolution of any curve about its axis, by putting for the radius $r$ the radius of curvature of the mirror at the point of incidence. Thus, suppose the mirror to be parabolical, and the latus rectum to be $L$, then for $r$ we must write $\frac{L}{2}$*, and the formula becomes

$$\frac{1}{v} + \frac{1}{u} = \frac{4}{L}.$$

The truth of the principle upon which this substitution is made will be seen at once, by considering that in the immediate neighbourhood of the point of incidence the curve and the circle of curvature may be supposed to coincide.

24. It has been observed (Art. 14) that, when a conical pencil of rays is incident on a spherical mirror, the reflected rays will not converge to or diverge from a point, unless we suppose the breadth of the pencil to be indefinitely small. There are however certain surfaces, on which if a pencil of rays of any magnitude be incident, the reflected rays will converge to or diverge from a point; such surfaces are said to be *aplanatic.*

25. *The surface formed by the revolution of a parabola about its axis, is aplanatic for rays incident parallel to its axis.*

By a property of the parabola, (Prop. II. Cor. 2, p. 165), the focal distance of any point and the line drawn through that point parallel to the axis make equal angles with the normal. Consequently, any ray incident parallel to the axis, will, after reflexion, pass through the focus, and therefore a pencil of rays, incident parallel to the axis of the surface

* The expression for the radius of curvature (page 354) is $2\frac{SP^2}{SY}$, and at the vertex this becomes $2AS$, which is equal to $\frac{L}{2}$. The same will be true for an ellipse or hyperbola.

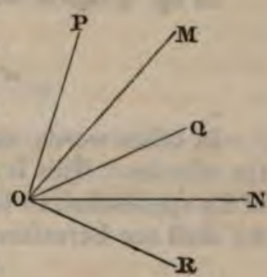
formed by the revolution of the parabola about its axis, will have the focus of the parabola for the focus of reflexion.

26. *The surface formed by the revolution of an ellipse about its major axis, is aplanatic for rays incident from one of its foci.*

By a property of the ellipse (Prop. II. p. 173), the focal distances of any point make equal angles with the normal at that point. Consequently any ray, incident from one focus, will, after reflexion, pass through the other, and therefore a pencil of rays, incident from one of the foci of the surface formed by the revolution of the ellipse about its major axis, will have the other focus for the focus of reflexion.

## COMBINED REFLEXIONS.

27. *To find the deviation of a ray of light, which is reflected at the surface of two plane mirrors, inclined to each other at a given angle, in the plane perpendicular to the line of intersection of the planes.*




Through any point $O$ draw $OM$, $ON$ parallel to the normals to the two mirrors. Let $PO$ be parallel to the incident ray; then, if we make $MOQ = MOP$, $OQ$ is parallel to the ray after reflexion at the first surface; and again, if we make $NOR = NOQ$, $OR$ will be parallel to the ray after reflexion at the second surface. Let $\alpha$ be the angle between the mirrors.

Then the deviation of the ray $= POR$

$$= POQ + QOR = 2MOQ + 2NOQ = 2MON = 2\alpha.$$

For simplicity of demonstration we have here represented only the normals of the surfaces and the directions of the rays; it may be useful however to represent the actual course of the ray when reflected at two surfaces, and this we proceed to do.

Let $AB$, $BC$ be sections of the two surfaces by the plane

in which the course of the ray $QRR'Q'$ lies; and let $QR$, $Q'R'$ intersect in $P$, then $QPR'$ is the angle of deviation which we wish to find. There are two cases to consider, according as

Fig. 1.

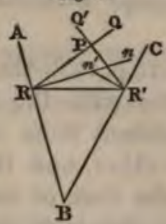
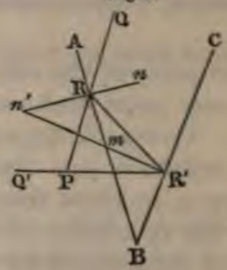



Fig. 2.




the normals $Rn$, $R'n'$ intersect between the planes (fig. 1), or not (fig. 2).

$$\text{In fig. 1,}\quad QPR' = PRR' + PR'R = 2\,(n'RR' + n'R'R)$$
$$= 2\,(\pi - Rn'R') = 2B.$$

$$\text{In fig. 2,}\quad QPR' = \pi - PRR' - PR'R = 2\left(\frac{\pi}{2} - mRR' - mR'R\right)$$
$$= 2\left(\frac{\pi}{2} - BmR'\right) = 2B.$$

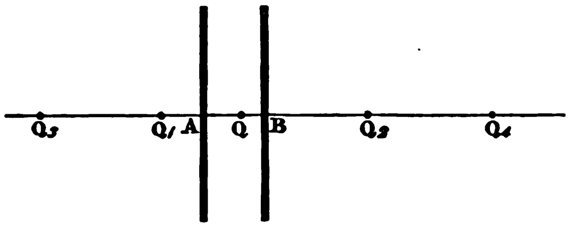
In other words, the deviation is twice the angle between the mirrors. This is an important proposition in consequence of its application to the construction of Hadley's Sextant, as we shall see hereafter.

28. *A luminous point is placed between two parallel plane mirrors; to find the position of the images, which will be formed by the successive reflexions of the rays at the surfaces of the two mirrors.*

Let $Q$ be the luminous point; through it draw $AQB$ perpendicular to the two mirrors, and produce it both ways.

Consider the rays which fall from $Q$ on the mirror $A$; if we make $AQ_1 = AQ$, an *image* of $Q$ will be formed at $Q_1$ (Art. 13.).

again, make $BQ_2 = BQ_1$, then the rays falling as from $Q_1$ on the mirror $B$ will form a second image $Q_2$; and if we




make $AQ_3 = AQ_2$, there will be a third image at $Q_3$, and so on.

Let $AQ = a$, $BQ = b$: then,

$$QQ_1 = 2a,$$

$$QQ_2 = BQ + BQ_2 = BQ + BQ_1 = 2BQ + QQ_1$$
$$= 2a + 2b,$$

$$QQ_3 = AQ + AQ_3 = AQ + AQ_2 = 2AQ + QQ_2$$
$$= 4a + 2b,$$

similarly, $QQ_4 = 2BQ + QQ_3 = 4a + 4b$;

and generally, $QQ_{2n} = 2na + 2nb$,

$$QQ_{2n+1} = (2n + 2)\,a + 2nb.$$

In like manner, if we consider the rays which fall from $Q$ on the mirror $B$, we shall have a series of images, $Q'_1$ $Q'_2$...... suppose, the position of which will be determined by the formulæ

$$QQ'_{2n} = 2na + 2nb,$$

$$QQ'_{2n+1} = 2na + (2n + 2)b.$$

Cor. If $a = b$, and $d$ be the distance between the mirrors, we shall have,

$$QQ_{2n} = 2nd,$$

$$QQ_{2n+1} = (2n + 1)\,d,$$

$$QQ'_{2n} = 2nd,$$

$$QQ'_{2n+1} = (2n + 1)\,d;$$

which formulæ are included in this one;

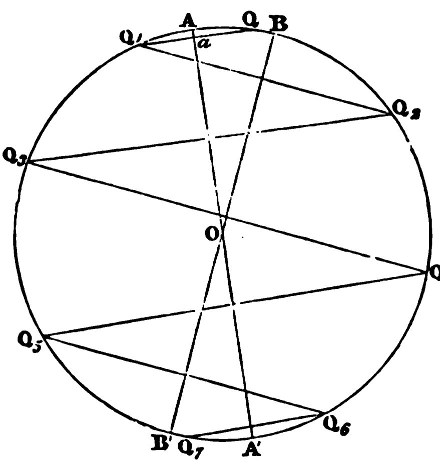
$$QQ_n = nd.$$

Hence the whole series of images will be equidistant from each other, and the distance between two consecutive images will be $d$.

29. *A luminous point is placed between two plane mirrors, inclined to each other at a given angle; to find the number and position of the images.*

Let $AOA'$, $BOB'$, be sections of the mirrors, made by a plane perpendicular to their line of intersection, and passing through the luminous point $Q$. With centre $O$ and radius $OQ$ describe a circle.

Consider first the rays which fall from $Q$ on the mirror $OA$; draw $QaQ_1$ perpendicular to $OA$ to meet the circle in $Q_1$, then $aQ_1 = aQ$, and therefore $Q_1$ is the image of $Q$: in like manner, if we draw $Q_1 Q_2$ perpendicular to $OB$ to meet the circle in $Q_2$, $Q_2$ will be the image formed by rays falling




from $Q_1$ on the mirror $OB$; and so on, the series of images $Q_1$ $Q_2$ $Q_3$... all lying on the circumference of the circle, which we have described.

Let the arc $AQ = \alpha$, $BQ = \beta$:

then $\quad QQ_1 = 2\alpha,$

$$QQ_2 = QB + BQ_2 = QB + BQ_1 = 2QB + QQ_1$$
$$= 2\alpha + 2\beta^*,$$

similarly, $\quad QQ_3 = 4\alpha + 2\beta;$

and generally, $QQ_{2n} = 2n\alpha + 2n\beta,$

$$QQ_{2n+1} = (2n + 2)\,\alpha + 2n\beta.$$

In like manner there will be a series of images, $Q'_1 Q'_2 \ldots$ suppose, formed by rays which at first fall on the mirror $OB$, the position of which will be determined by the formulæ,

$$QQ'_{2n} = 2n\alpha + 2n\beta,$$
$$QQ'_{2n+1} = 2n\alpha + (2n+2)\,\beta.$$

To determine the number of the images, we observe that reflexion will be repeated until one of the images falls at the back of the mirrors, as $Q_7$ in the figure; that is to say, if $Q_p$ be the last of the series of images, $Q_1\, Q_2 \ldots$, $p$ must be such that $AQ_p$ is greater than $\pi$ if $p$ be even, and $BQ_p$ greater than $\pi$ if $p$ be odd.

Suppose $p$ to be even and $= 2n$, then $AQ_p = \alpha + QQ_{2n}$, and we must have

$$\alpha + 2n\alpha + 2n\beta > \pi,$$

$$\text{or } p > \frac{\pi - \alpha}{\alpha + \beta}.$$

Suppose $p$ to be odd, and $= 2n + 1$,

$$\text{then } BQ = \beta + QQ_{2n+1},$$

and we must have

$$\beta + (2n + 2)\,\alpha + 2n\beta > \pi.$$

$$\text{or } p > \frac{\pi - \alpha}{\alpha + \beta}.$$

* It is not difficult to see that $QQ_2$ measures the deviation after two reflexions; hence this formula is identical with that of Art. 27.

Hence, whether $p$ be even or odd, it will be the whole number next greater than $\dfrac{\pi - \alpha}{\alpha + \beta}$.

Similarly, the number of images formed by rays which fall from $Q$ on the mirror $OB$ will be the whole number next greater than $\dfrac{\pi - \beta}{\alpha + \beta}$.

COR. If $\alpha + \beta$, the angle between the mirrors, be an integral part of $\pi$, the number of each set of images will be $\dfrac{\pi}{\alpha + \beta}$, for this will be a whole number, and $\dfrac{\alpha}{\alpha + \beta}$, $\dfrac{\beta}{\alpha + \beta}$ will be proper fractions.

Suppose, for instance, $\alpha + \beta = \dfrac{\pi}{3}$, then each set will consist of 3 images. If we suppose the point $Q$ to be symmetrically situated with respect to the mirrors, the last of the two series of images will coincide, and there will be found altogether 5 images, so that the object and images will be in the angular points of a regular hexagon. And even if the point $Q$ be not situated symmetrically with respect to the mirrors, the object and images will still lie in the angular points of a hexagon, though not a regular hexagon. For let $Q_3$ and $Q'_3$ be the last of the two sets of images as in the proposition; then we have, as above,

$$QQ_3 = 4\alpha + 2\beta, \quad QQ'_3 = 4\beta + 2\alpha,$$

$$\therefore\ QQ_3 + QQ'_3 = 6\alpha + 6\beta = 2\pi, \text{ since } \alpha + \beta = \frac{\pi}{3}.$$

Hence $Q_3$ and $Q'_3$ coincide, and therefore there will be in reality only 5 distinct images.

The preceding proposition contains the principle of the toy called the *kaleidoscope*.

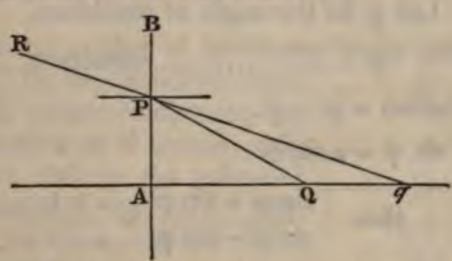
## ON REFRACTION AT A SINGLE SURFACE.

30. We shall now give investigations of cases of refraction, analogous to those of reflexion already considered. The figures will be drawn in general on the supposition of

light being refracted either from vacuum into a medium, or from a rarer into a denser medium; in other words, $\mu$ will be supposed to be greater than unity, or the refracted ray to be bent *towards* the normal, unless the contrary is stated: the formulæ obtained will however be generally true.

### I. *A plane Surface.*

31. *Diverging rays are incident upon a plane refracting surface; to find the geometrical focus.*




Let $Q$ be the focus of incidence; $QA$ a normal to the surface $AB$ of the refracting medium; $PR$ the refracted ray which produced backwards meets $QA$ in $q$; then the ultimate position of $q$, when $P$ moves up to $A$, is the geometrical focus.

Let $AQ = u$, $Aq = v$, $AQP = \phi$, $AqP = \phi'$;

$$\text{then, } \sin\phi = \mu \sin\phi',$$

$$\text{but } \sin\phi = \frac{AP}{PQ}, \ \sin\phi' = \frac{AP}{Pq},$$

$$\therefore Pq = \mu PQ;$$

and this, being always true, will be true when $P$ has moved up to $A$;

$$\therefore v = \mu u.$$

Cor. When rays proceed from a denser into a rarer medium, $\mu$ is less than 1, and therefore $q$ is nearer to the surface than $Q$. Hence when we look at an object which is under water, it will appear to be nearer to the surface of the water than it really is; for the rays which proceed from $Q$, when they emerge into air, proceed as if from $q$.

Hence also, a pole thrust partly into water will appear discontinuous at the surface of the water, and the part submerged will seem to be bent upwards; for the rays from each point of the part submerged will come to the eye, as if from a point nearer to the surface than the point itself.

32. *When a ray of light passes out of one medium into another, as the angle of incidence increases, the angle of deviation also increases.*

Let $\phi$ be the angle of incidence,

$\phi'$ ................. refraction,

then the deviation $= \phi - \phi'$.

Also, let $\sin\phi = \mu \sin\phi'$;

$$\text{then} \quad \frac{\sin\phi + \sin\phi'}{\sin\phi - \sin\phi'} = \frac{\mu + 1}{\mu - 1},$$

$$\text{or} \quad \frac{\sin\frac{\phi+\phi'}{2}\cos\frac{\phi-\phi'}{2}}{\cos\frac{\phi+\phi'}{2}\sin\frac{\phi-\phi'}{2}} = \frac{\mu+1}{\mu-1};$$

$$\therefore \tan\frac{\phi-\phi'}{2} = \frac{\mu-1}{\mu+1}\tan\frac{\phi+\phi'}{2}.$$

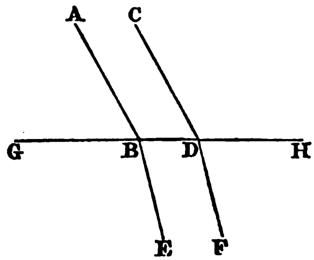
Now as $\phi$ increases it is manifest that $\phi'$ increases, because $\sin\phi' = \frac{1}{\mu}\sin\phi$; $\therefore \frac{\phi+\phi'}{2}$ $\left(\text{and } \therefore \tan\frac{\phi+\phi'}{2}\right)$ increases; and $\therefore \tan\frac{\phi-\phi'}{2}$ increases, or the deviation $\phi - \phi'$ increases.

This result is true whether $\mu$ is greater or less than unity: that is, whether the refraction takes place from a rarer into a denser medium, or from a denser into a rarer.

33. *Parallel rays refracted at a plane surface continue parallel.*

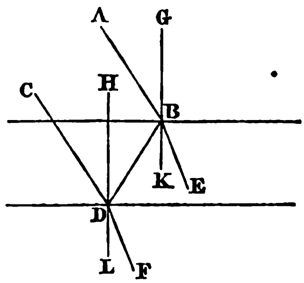
(1) Let the angles of incidence be in the same plane.

Let $GH$ be the refracting surface; $AB$, $CD$ two parallel incident rays, $BE$, $DF$ the corresponding refracted rays.




Then $ABG = CDG$, that is, the angles of incidence are equal; consequently the angles of refraction must be equal; and therefore $EBD$, $FDH$, which are the complements of the angles of refraction, are equal; that is, $BE$ is parallel to $DF$.

(2) Let the angles of incidence be in different planes. Let $AB$, $CD$ be two incident rays; $GBK$, $HDL$ perpendiculars to the refracting surface at $B$ and $D$; join $BD$; and let $BE$ be the refracted ray corresponding to $AB$. Also let $DF$ be the intersection of the planes $EBD$, $CDH$.




Then $GBK$, $HDL$ being perpendicular to the same plane are parallel (Euc. XI. 6); and $AB$ is parallel to $CD$ by hypothesis; therefore the angles $ABG$, $CDH$ are equal, and therefore the angles of refraction are equal.

Again, since $AB$, $BG$ are respectively parallel to $CD$, $DH$, the planes $ABG$, $CDH$ are parallel (Euc. XI. 15); and they are intersected by the plane $EBDF$, therefore $BE$ is parallel to $DF$: therefore the angles $EBK$, $FDL$ are equal; but $EBK$ is the angle of refraction for the ray $AB$, therefore $FDL$ is equal to the angle of refraction for $CD$. And $FDL$ is in the same plane with the angle of incidence $CDH$; therefore the refracted ray corresponding to $CD$ is $DF$, which has before been shewn to be parallel to $BE$.

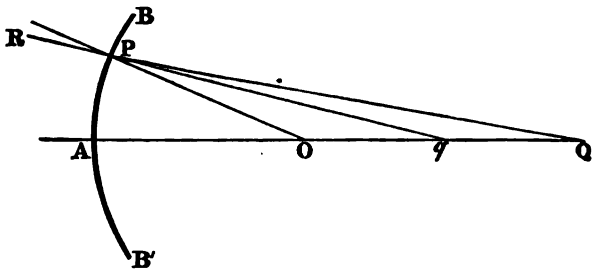
## II. *A Spherical Surface.*

We shall give the investigation of the position of the geometrical focus, for the case of rays diverging upon a concave surface; the student will be able to apply the same

method to all other cases, which will, however, be included in the formula which we shall obtain, by having regard to the convention respecting the negative sign explained in Art. 17.

34. *Diverging rays are incident upon a concave spherical refracting surface: to find the geometrical focus.*

Let $BAB'$ be a section of the refracting surface, made by a plane passing through the centre $O$ of the sphere, and the




focus of incidence $Q$. Let $QP$ be any incident ray, $PR$ the corresponding refracted ray, which produced backwards meets $QOA$ in $q$. Then the ultimate position of $q$, when $P$ moves up to $A$, will be the geometrical focus.

Join $PO$, and let

$$OPQ = \phi,\quad OPq = \phi',\quad AQ = u,\quad Aq = v,\quad AO = r;$$

then, in the triangle $OPq$,

$$\frac{Oq}{Pq} = \frac{\sin\phi'}{\sin POq},$$

and in the triangle $OPQ$,

$$\frac{OQ}{PQ} = \frac{\sin\phi}{\sin POQ} = \frac{\mu\sin\phi'}{\sin POq};$$

$$\therefore \frac{\mu Oq}{Pq} = \frac{OQ}{PQ};$$

and this, being always true, will be true when $P$ has moved up to $A$, in which case

$$PQ = u,\quad Pq = v,\quad Oq = v - r;$$

$$\therefore \frac{\mu(v-r)}{v} = \frac{u-r}{u},$$

$$\text{or } \frac{\mu}{r} - \frac{\mu}{v} = \frac{1}{r} - \frac{1}{u},$$

$$\text{or } \frac{\mu}{v} - \frac{1}{u} = \frac{\mu-1}{r}.$$

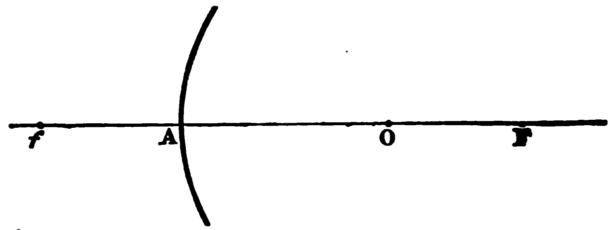
OBS. It will be seen, that this formula reduces itself to that for reflexion by putting $\mu = -1$; and this is the case with all formulæ for refraction.

COR. If $u = \infty$, or the incident rays are parallel, $v = \dfrac{\mu}{\mu-1} r$, a result which may also be easily obtained by direct investigation.

The focus of refraction for parallel rays may be termed the *principal* focus. Hence the distance of the principal focus, for rays passing from vacuum into a medium, from the surface, is $\dfrac{\mu}{\mu-1} r$. If the rays had passed from the medium into vacuum, the distance would have been $\dfrac{\frac{1}{\mu}}{\frac{1}{\mu}-1} r = \dfrac{r}{\mu-1}$, (without regard to sign;) and the distance from the centre would be $r + \dfrac{r}{\mu-1} = \dfrac{\mu}{\mu-1} r$; hence the distance of one of the principal foci spoken of from the surface is equal to that of the other from the centre.

35. The relative positions of the conjugate foci are not so worthy of notice in the case of refraction at a spherical surface, as in that of reflexion, because in practice we are seldom or never concerned with refraction at a single surface: nevertheless, the student will find the following proposition worthy of attention, as tending to familiarize him with investigations of the same kind.

36. *To trace the corresponding positions of the conjugate foci.*



Suppose the surface to be concave, then we have

$$\frac{\mu}{v} - \frac{1}{u} = \frac{\mu - 1}{r}.$$

Since the difference of $\frac{\mu}{v}$ and $\frac{1}{u}$ is constant, as one of them increases, the other must also increase, and the same will be true of $v$ and $u$; hence $Q$ and $q$ always move in the same direction.

(1) Let $Q$ be at an infinite distance to the right of the mirror, or the incident rays parallel; then $v = \frac{\mu}{\mu - 1} r = AF$, suppose, and $q$ is at $F$.

(2) Let $Q$ move towards $O$; then $q$ also moves towards $O$, and at $O$ they coincide, because when $u = r$, $v = r$.

(3) Let $Q$ move from $O$ towards $A$; then $q$ also moves towards $A$, and when $Q$ arrives at $A$, $q$ arrives there also, because when $u = 0$, $v = 0$.

(4) Let $Q$ move to the left of $A$; then $q$ also moves to the left of $A$, and when $Q$ has reached a point $f$, such that $Af = \frac{r}{\mu - 1}$, $q$ is at an infinite distance, or the refracted rays parallel, because when $u = -\frac{r}{\mu - 1}$, $v = \infty$.

(5) Let $Q$ move to the left of $f$; then $q$ moves up from an infinite distance to the right of $A$, and when $Q$ has attained to an infinite distance, $q$ is at $F$, because when $u = \infty$, $v = \dfrac{\mu}{\mu - 1} r$.

$Q$ and $q$ are now in the same positions as at first, and therefore we have traced all their corresponding positions.

[37. The preceding Articles contain a complete investigation of the form of the refracted pencil for the case of incidence on a spherical surface. We shall however subjoin the enunciation and independent geometrical proof of two or three theorems, which may readily be deduced from our fundamental formula.

38. *When a small pencil of parallel rays is directly incident upon a spherical refracting surface, the distance of the geometrical focus from the surface is to its distance from the centre, as the sine of the angle of incidence to the sine of the angle of refraction.*

There will be four different cases,

1 Concave surface; rays from rarer into denser medium.
2. ............................... denser ... rarer ......
3. Convex surface; ............ rarer ... denser ......
4. ............................... denser ... rarer ......

Fig. 1.

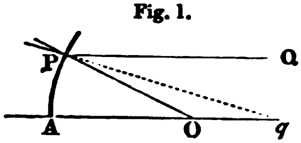



Fig. 2.

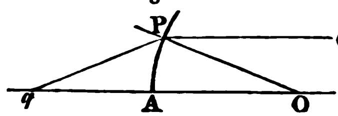



Fig. 3.

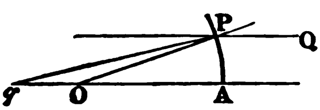
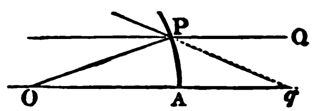



Fig. 4.




Let $QP$ be a ray incident very near the axis, $O$ the

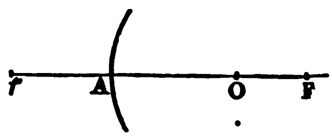
centre of the spherical surface, $Pq$ the direction of the refracted ray. Then in each of the figures we have

$$Pq : Oq :: \sin POq : \sin OPq$$
$$:: \sin POA : \sin OPq$$
$$:: \sin OPQ : \sin OPq$$

:: sine of the angle of incidence : sine of the angle of refraction.

Now let $P$ be indefinitely near to $A$, then $q$ becomes the geometrical focus, and $Pq$ becomes $Aq$, the distance of $q$ from the surface, and $Oq$ its distance from the centre. Hence the proposition enunciated is true.

39. *When parallel rays are directly incident in opposite directions upon a spherical refracting surface, the distance of one of the foci of refracted rays from the surface is equal to the distance of the other from the centre.*




Let $F$ be the focus when the rays pass from the rarer medium into the denser, $f$ when they pass from the denser into the rarer. Then, by the preceding proposition,

$$FA : FO :: \mu : 1,$$
$$\therefore AO : FO :: \mu - 1 : 1.$$

In like manner

$$fA : fO :: \frac{1}{\mu} : 1 \text{ (Art. 9, Cor.)}$$
$$:: 1 : \mu,$$
$$\therefore fA : AO :: 1 : \mu - 1.$$

Hence $FO = fA$,

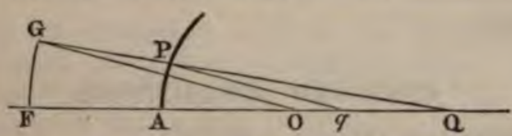
and $fO = fA + AO = FO + AO = FA$.

40. *When a small pencil of diverging or converging rays is incident directly upon a spherical refracting surface, the distance of the focus of incident rays from the principal focus of rays coming in the opposite direction, is to its distance from the*

*centre, as its distance from the surface, to its distance from the geometrical focus.*

The proposition admits of *eight* cases, since the rays may either *diverge* or *converge* upon a *concave* or *convex* surface, and may pass from a *rarer* into a *denser* or from a *denser* into a *rarer* medium. We shall give the demonstration in two cases; a similar method will apply in all the rest.

(1) Let the rays diverge upon a concave surface, and pass from a rarer medium into a denser.




Let $Q$ be the focus of incidence, $O$ the centre of the surface, $QP$ an incident ray, $F$ the principal focus of rays coming in the opposite direction.

With centre $O$ describe an arc of a circle $FG$, meeting the ray $QP$ produced in $G$; join $GO$, and draw $Pq$ parallel to $GO$.

Then since $G$ is the principal focus of a pencil of rays incident, (from the *left*, as the figure is drawn,) parallel to $GO$, any ray of such pencil will after refraction proceed from $G$. Hence, conversely, any ray incident from the *right* and proceeding towards $G$ will after refraction become parallel to $GO$. Hence $qP$ which is parallel to $GO$ will be the direction after refraction of the ray $QP$ which is proceeding towards $G$. Therefore when $P$ approaches indefinitely near to $A$, $q$ will be ultimately the geometrical focus.

Now by similar triangles $QGO$, $QPq$,

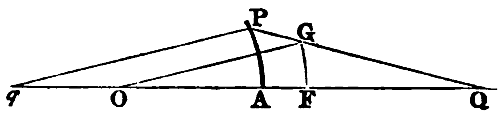
$$QG : QO :: QP : Qq,$$

$$\text{or } QF : QO :: QA : Qq.$$

(2) Let the rays diverge upon a convex surface, and pass from a rarer medium into a denser.

The construction will be similar, the difference will be

that $Q$ and $F$ are on the same side of the surface, and $q$ on




the opposite side. We shall have, as before,

$$QG : QO :: QP : Qq,$$

$$\text{or } QF : QO :: QA : Qq.$$

The proposition may be proved in like manner in all the other cases.]

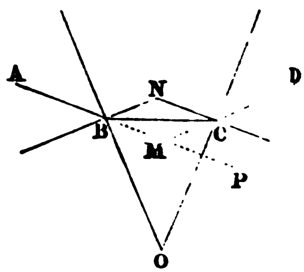
## COMBINED REFRACTIONS.

41. *Refraction through a prism.*

DEF. A prism is a portion of a refracting medium bounded by two planes, which terminate in a common line called the *edge* of the prism.

DEF. The refracting angle of a prism is the angle of inclination of the two bounding planes.

*A ray of light is refracted through a prism, in a plane perpendicular to its edge; to find the direction of the refracted ray.*




Let the plane of the paper be that in which the refraction takes place; and let $ABCD$ be the course of the ray through the prism.

Let $\phi$, $\phi'$ be the angles of incidence and refraction at the first surface,

$\psi'$, $\psi$, those at the second surface,

$i$ the angle of the prism,

$D$ the whole deviation.

The deviation at the first refraction is $\phi - \phi'$, and at the

second $\psi - \psi'$; and, as we have drawn the figure*, the two deviations evidently take place in the same directions;

$$\text{hence, } D = \phi - \phi' + \psi - \psi',$$

and we have, $\psi' + \phi' = 180^\circ - BNC = i$ ............ (1);

$$\therefore D = \phi + \psi - i \text{ ............ (2).}$$

From the equations (1) and (2), together with the relations

$$\sin\phi = \mu \sin\phi',$$
$$\sin\psi = \mu \sin\psi',$$

$\phi' \psi \psi'$ may be eliminated, and we shall then have $D$ expressed in terms of the given quantities $\phi$ and $i$.

If we had drawn the incident ray $AB$ on the other side of the normal to the surface of the prism, the deviation $D$ would have been the *difference* of the two deviations $\phi - \phi'$ and $\psi - \psi'$†.

* We might have drawn two figures as in the case of reflexion, (page 482), but this does not appear to be necessary. It may be remarked also that the student may, if he pleases, treat this proposition in a manner analogous to the first demonstration of Art. 27, page 481.

† To determine under what circumstances the deviation will be a minimum.

We have the equations

$$\sin\phi = \mu\sin\phi' \qquad (1)$$
$$\sin\psi = \mu\sin\psi' \qquad (2)$$
$$\phi' + \psi' = i \qquad (3)$$
$$D = \phi + \psi - i \qquad (4);$$

from (2) and (3),

$$\sin\psi = \mu\sin(i - \phi') = \mu\sin i\cos\phi' - \mu\cos i\sin\phi',$$
$$\therefore \sin\psi + \cos i\sin\phi = \mu\sin i\cos\phi',$$

also

$$\sin i\sin\phi = \mu\sin i\sin\phi';$$

$\therefore$ adding the squares of these equations,

$$\mu^2\sin^2 i = \sin^2\psi + \sin^2\phi + 2\cos i\sin\psi\sin\phi$$
$$= \frac{1-\cos 2\psi}{2} + \frac{1-\cos 2\phi}{2} + 2\cos i\sin\psi\sin\phi$$
$$= 1 - \cos(\psi+\phi)\cos(\psi-\phi) + \cos i\,\{\cos(\psi-\phi) - \cos(\psi+\phi)\}$$
$$\therefore (\mu^2-1)\sin^2 i = \cos^2 i + \cos i\,\{\cos(\psi-\phi) - \cos(\psi+\phi)\} - \cos(\psi+\phi)\cos(\psi-\phi)$$
$$= \{\cos i + \cos(\psi-\phi)\}\,\{\cos i - \cos(\psi+\phi)\}.$$

Now it is evident that $D$ diminishes as $\cos i - \cos(\psi+\phi)$ diminishes, and therefore as $\cos i + \cos(\psi-\phi)$ increases. But this last quantity will have its greatest value when $\cos(\psi-\phi) = 1$, or $\phi = \psi$. Hence the deviation will be a minimum when the angles of incidence and emergence are equal.

Cor. Let $i$ be very small, $\phi'$ and $\psi'$ both very small, and therefore $\phi$ and $\psi$. Then, if the angles be expressed in the circular measure,

$$\phi = \mu\phi', \text{ and } \psi = \mu\psi', \text{ nearly, (Art. 57, page 155),}$$

$$\therefore\ D = \mu(\phi' + \psi') - i = (\mu - 1)\,i.$$

This is a remarkable result, shewing that for prisms of very small angles the deviation of a ray falling nearly perpendicularly upon the prism is independent of the direction of incidence.

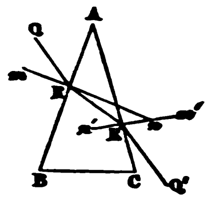
42. *When a ray of light passes through a prism, which is denser than the surrounding medium, in a plane perpendicular to its edge, the deviation of the ray is always from the refracting angle, or towards the thicker part of the prism.*

There are three cases to be considered.

(1) Let $QRR'Q'$ be the course of the ray of light passing through the prism $BAC$; and let the angles $ARR'$, $AR'R$ be both acute. Then it is evident, that in this case the deviation is *from* the angle of the prism at *each* refraction, and therefore the *total* deviation is *from* the angle.


(2) Let the angle $ARR'$ be a right angle; then there is no deviation at the first refraction, and the deviation is *from* the angle at the second; therefore the *total* deviation is *from* the angle. And the same would be true if we supposed $AR'R$ to be a right angle.


(3) Let the angle $ARR'$ be an obtuse angle; then $QR$ must lie between $AR$ and the normal $mR$, and the first deviation will be *towards A*, the second will be *from* it, and it will be necessary to shew that the second deviation is




greater than the first. Now the angle $RR'n'$, which the ray $RR'$ makes with the normal to the second surface $n'R'm'$, is greater than $R'Rn$ which the same ray makes with the normal to the first surface, $RR'n'$ being the exterior angle of the triangle $RR'n$; consequently the deviation at the second surface is greater than that at the first, by Art. 32, taken in conjunction with the first of the laws cited in Art. 9. Hence in all cases the deviation is *from* the refracting angle of the prism.

43. *To construct a prism, such that no ray shall be able to pass through it.*

Suppose the angle of the prism to be such, that a ray being incident very nearly parallel to one face, emerges very nearly parallel to the other; then it is clear, that if the refracting angle be greater than that determined by this condition, no ray will be able to pass through, because every ray which enters the prism will be incident on the second surface at an angle greater than the critical angle, and will therefore be totally reflected within the prism.

But this condition gives us, that

$$\phi = 90^\circ, \qquad \psi = 90^\circ;$$

$$\therefore \sin\phi' = \frac{1}{\mu}, \qquad \sin\psi' = \frac{1}{\mu}.$$

$$\text{Hence, } i = \phi' + \psi' = 2\phi',$$

$$\text{or } \sin\frac{i}{2} = \frac{1}{\mu}.$$

The value of this angle for glass is about $83^\circ$.

44. *Refraction through a lens.*

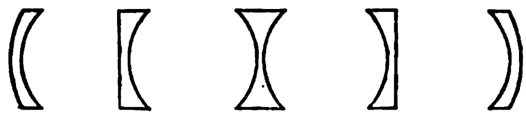
Def. A lens is a portion of a refracting medium, bounded by two spherical surfaces. The line joining the

centres of the spherical surfaces is called the *axis* of the lens.

Under this general definition, we include the case of one of the surfaces of the lens being plane, since a plane may be regarded as a sphere of infinite radius.

Lenses are of various kinds, and receive different names according to the form of the surfaces which bound them. The rule for assigning the proper designation to a lens is this: first look at the face of the lens on which the light is incident, then reverse the lens and look at the other face; combine the names, which indicate the form of the surfaces successively presented to the eye, and the combination will give the name of the lens.

If we suppose, as heretofore, that light proceeds across the paper from right to left, then the names and forms of the various lenses will be as under;



I. II. III. IV. V.

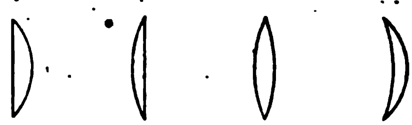
I. *Concavo-convex,*

II. *Concavo-plane,*

III. *Concavo-concave* or *double concave,*

IV. *Plano-concave,*

V. *Convexo-concave,*



VI. VII. VIII. IX.

VI. *Convexo-plane,*

VII. *Plano-convex,*

VIII. *Convexo-convex* or *double convex.*

IX. A lens such as that represented by fig. IX., according to the rule above given, would be a convexo-concave; but

having very different properties from a lens of that description, it is distinguished by the name of *meniscus*.

45. *To find the geometrical focus of a small pencil of diverging rays refracted through a concavo-convex lens, the thickness of which is neglected.*

Let $r$, $s$ be the radii of the surfaces of the lens.
$u$ the distance from the lens of the focus of incidence.
$V$ .......................... of the focus after refraction through the first surface.
$v$ .......................... through the second.

Then, by Art. 34, we have

$$\frac{\mu}{V} - \frac{1}{u} = \frac{\mu - 1}{r} \quad (1).$$

Now the rays fall upon the second surface, as if they came from the geometrical focus, the position of which is determined by equation (1); but they emerge from the medium of which the lens is formed into vacuum, therefore in adapting the formula to this case we must write $\frac{1}{\mu}$ instead of $\mu$, (Art. 9. Cor.); hence, for the second surface,

$$\frac{1}{\mu}\frac{1}{v} - \frac{1}{V} = \frac{\frac{1}{\mu} - 1}{s},$$

$$\text{or } \frac{1}{v} - \frac{\mu}{V} = -\frac{\mu - 1}{s} \quad (2).$$

Adding (1) and (2), we have

$$\frac{1}{v} - \frac{1}{u} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right);$$

which is the formula required, and gives $v$ when $u$ is known.

46. The investigation is quite similar for the case of converging or parallel rays, or for any other species of lens.

We have taken the particular case of rays diverging on a concavo-convex lens, because in it all lines are measured in the same direction, namely, towards the source of light; the formula for any other lens may be deduced, by attending to the convention already explained respecting the *minus* sign.

47. If we make $u = \infty$, we obtain $\frac{1}{v} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right)$; the point thus determined is called the *principal focus*, or more briefly, the *focus*, of the lens. The distance of this point from the lens is called its *focal length*, and if we denote it by $f$, we shall have

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right).$$

On examining this expression it will be seen, that the first five lenses, enumerated in Art. 44, have their focal lengths positive, and the last four negative; that is to say, if parallel rays be incident upon either of the first five, they will *diverge* after refraction, if upon either of the last four, they will *converge*. Hence, lenses divide themselves naturally into two classes, which may be distinguished by the algebraical sign of $f$; those for which $f$ is positive may be described in general as *concave* lenses, and those for which it is negative as *convex*. So far as the purpose of this treatise is concerned, we shall distinguish the lenses used in optical instruments only by this general character; when greater refinement is sought, there are other considerations which render the use of particular lenses in particular cases desirable.

48. It is manifest that a lens will produce the same effect, whichever face is presented to the incident light, since the formula which determines the position of the geometrical focus is not altered by supposing the face changed. Also, it appears, that a lens has two principal foci; for whichever face of the lens is exposed to parallel rays, the rays will have the same geometrical focus, and therefore a point on the axis of the lens, and at a distance $f$ from it on either side of the lens, may be called its principal focus, and parallel rays will have one focus or the other as their geometrical focus, according as they fall on one face of the lens or the other.

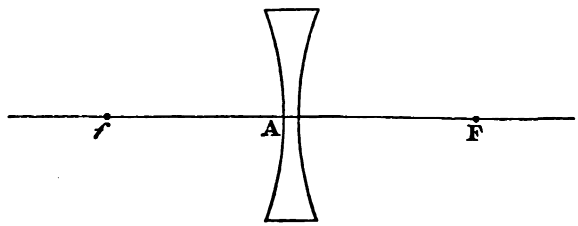
49. *To trace the corresponding positions of the conjugate foci, for a concave lens.*

The formula which determines the position of $q$ is,

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}.$$

Since the difference of $\frac{1}{v}$ and $\frac{1}{u}$ is constant, $\frac{1}{v}$ and $\frac{1}{u}$ must increase and decrease together, and the same will be true of $v$ and $u$; hence $Q$ and $q$ move always in the same direction.

Let $F$ and $f$ be the foci of the lens, which are at a distance $f$ from $A$. Then,




(1) Let $Q$ be at an infinite distance to the right of $A$, or the incident rays be parallel; then $q$ is at $F$, because when $u = \infty$, $v = f$, and the rays are divergent.

(2) Let $Q$ move towards $A$; then $q$ moves in the same direction, and at $A$ they coincide, because when $u = 0$, $v = 0$.

(3) Let $Q$ move to the left of $A$; then $q$ also moves to the left of $A$, and when $Q$ has reached $f$, $q$ is at an infinite distance, because when $u = -f$, $v = \infty$, and the refracted rays are parallel.

(4) Let $Q$ move to the left of $f$; then $q$ moves up in the same direction from an infinite distance on the right of $A$, and when $Q$ is at an infinite distance, $q$ is at $F$, because when $u = \infty$, $v = f$.

$Q$ and $q$ are now in the same positions as at first, and we have therefore traced all their corresponding positions.

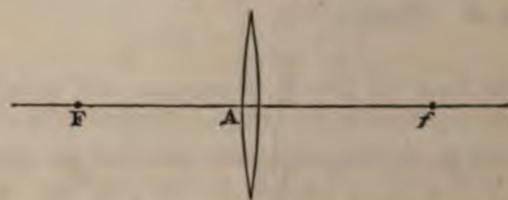
In the figure we have represented a double concave lens, but the same investigation is applicable to all lenses for which

$f$ is positive. On account of the importance to the student of perfect familiarity with the relative positions of the conjugate foci, we shall consider at length the case of a convex lens, or in general of a lens for which $f$ is negative.

50. *To trace the corresponding positions of the conjugate foci, for a convex lens.*

The formula which determines the position of $q$ is,

$$\frac{1}{v} - \frac{1}{u} = -\frac{1}{f}.$$




$Q$ and $q$ move in the same direction, as before. Let $F$ and $f$ be the foci of the lens, which are at a distance $f$ from $A$. Then,

(1) Let $Q$ be at an infinite distance to the right of $A$, or the incident rays parallel; then $q$ is at $F$, because when $u = \infty$, $v = -f$, and the refracted rays are convergent.

(2) Let $Q$ move towards $f$; then $q$ moves to the left of $F$, and when $Q$ has reached $f$, $q$ is at an infinite distance, because when $u = f$, $v = \infty$, and the refracted rays are parallel.

(3) Let $Q$ move from $f$ towards $A$; then $q$ moves up from an infinite distance on the right of the lens towards $A$, and at $A$ they coincide, because when $u = 0$, $v = 0$.

(4) Let $Q$ move to the left of $A$; then $q$ also moves to the left of $A$, and when $Q$ is at an infinite distance, $q$ is at $F$, because when $u = \infty$, $v = -f$.

$Q$ and $q$ are now in the same positions as at first, and therefore we have traced all their corresponding positions.

51. In examining the preceding investigations, it will be seen, that the concave lens always diminishes the conver-

gency or increases the divergency of rays, and that the convex lens has exactly the reverse effect. This very important property of lenses admits however of simple direct proof, as in the following proposition.

52. *A concave lens diminishes the convergency or increases the divergency of a pencil of rays, and a convex lens produces the opposite effect.*

For the concave lens,

$$\frac{1}{v} = \frac{1}{u} + \frac{1}{f}.$$

Suppose $u$ positive, or the incident rays divergent, then $\frac{1}{v}$ is greater than $\frac{1}{u}$, or $v$ less than $u$; therefore the geometrical focus is nearer to the lens than the focus of incidence, or the refracted rays are more divergent. If $u$ is negative, or the incident rays convergent, $v$ is either positive or a greater negative quantity than $u$; therefore the refracted rays are either divergent, or less convergent than the incident.

For the convex lens,

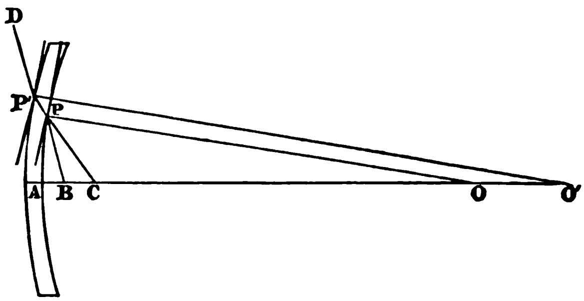
$$\frac{1}{v} = \frac{1}{u} - \frac{1}{f}.$$

From the discussion of which equation, the opposite conclusions to the above are deduced. Hence the proposition is true.

53. When reflexion takes place at a spherical surface, it is evident that a ray which passes through the centre of the sphere suffers no deviation, because it falls upon the surface in a direction coinciding with the normal at that point: there is a similar point on the axis of a lens, through which, if the direction of a ray during its course through the lens pass, it will emerge parallel to the direction of incidence, and therefore if we neglect the thickness of the lens, will suffer no deviation; this point by analogy is called the *centre of the lens.*

54. *To find the centre of a lens.*

Let the lens be concavo-convex. $OO'$ the centres of the first and second surfaces respectively.




Draw any line $OP$ to the first surface, and $O'P'$, parallel to $OP$, to the second surface; join $PP'$; and let $BPP'D$ be the course of a ray through the lens; produce $PP'$ to meet the axis in $C$.

Then, since $OP$, $O'P'$ are parallel, the planes perpendicular to them are parallel, and therefore the ray $BPP'D$ passes as through a medium bounded by parallel surfaces, and $P'D$ is parallel to $BP$.

Again, by similar triangles, $CPO$, $CP'O'$,

$$\frac{CO}{PO} = \frac{CO'}{P'O'},$$

or, $$\frac{r - AC}{r} = \frac{s - t - AC}{s},$$ where $t$ is the thickness of the lens;

$$\therefore AC = \frac{rt}{s - r}.$$

This formula determines the position of $C$, the centre of the lens, which, as we see, depends on the form of the lens only.

55. If the surfaces of the lenses have their convexities turned in opposite directions, that is, if $r$ and $s$ have different algebraical signs, the distance $AC$ is very small when $t$ is very small; and hence, in the case of double convex or double

concave lenses, which are very thin, we may consider in practice that the centre coincides with the middle point $A$ of the lens, and therefore that a ray passing through $A$ suffers no deviation.

But the centre of a concavo-convex or of a meniscus lens may be at a considerable distance from the lens, if $r$ and $s$ be nearly equal.

Let us investigate the condition of the centre lying within the substance of the lens. It is evident that the condition is this, that $AC$ (as we have measured it) must be a negative quantity and numerically less than $t$; and this we may express by putting for $AC$ the expression $-t\left(1-\dfrac{1}{x}\right)$, where $x$ is a positive quantity greater than unity. Hence we have

$$-t\left(1-\frac{1}{x}\right)=\frac{rt}{s-r},$$

$$\text{or}\quad -s+r+\frac{s-r}{x}=r,$$

$$\therefore\ \frac{s-r}{x}=s,$$

$$\text{or}\quad x=1-\frac{r}{s};$$

now $x$ is a positive quantity greater than 1, therefore $\dfrac{r}{s}$ must be a negative quantity, or $r$ and $s$ must have opposite signs; consequently for the double convex and double concave the centre lies within the substance of the lens. In the case of the plano-convex or plano-concave the centre lies upon the curved surface.

56. Sometimes two lenses are placed on the same axis, and very near to each other, so as to serve the purpose of one lens: the focal length of the lens, which would produce the same effect on a pencil of rays as the two together, is called the *focal length of the combination*.

57. *To find the focal length of a combination of two lenses, omitting their thickness.*

Let $ff'$ be the focal lengths of the lenses, $F$ that of the combination.

Suppose rays to diverge from a point at distance $u$ from the first lens, and let $V$ be the distance of the geometrical focus; from this focus rays diverge upon the second lens, let $v$ be the distance of the geometrical focus after this second refraction.

Then we have,

$$\frac{1}{V} - \frac{1}{u} = \frac{1}{f},$$

$$\text{and } \frac{1}{v} - \frac{1}{V} = \frac{1}{f'};$$

adding these equations,

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f} + \frac{1}{f'}.$$

But if the rays diverged upon a lens of focal length $F$, the formula would be

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{F}.$$

Hence, in order that this lens may be equivalent to the combination, we must have

$$\frac{1}{F} = \frac{1}{f} + \frac{1}{f'}.$$

If the first lens be concave, and the second convex, we should have in like manner,

$$\frac{1}{F} = \frac{1}{f} - \frac{1}{f'};$$

and so in other cases.

A similar investigation is applicable to three or more lenses.

58. *To find by experiment the focal length of a lens.*

First suppose the lens convex; and let $Q$ be a luminous point upon its axis, $q$ the image of $Q$ formed upon the opposite side of the lens; then we will first shew (by the method given in p. 45, note) that when $Q$ is so adjusted that the distance $Qq$ is the least possible, $Qq$ is equal to four times the focal length of the lens.

In the general formula $\frac{1}{v} - \frac{1}{u} = -\frac{1}{f}$,

$$\text{let} \quad u - v = x;$$

$$\therefore fx = -uv = -u(u-x);$$

$$\therefore u^2 - ux + \frac{x^2}{4} = \frac{x^2}{4} - fx,$$

$$u = \frac{x}{2} \pm \sqrt{\frac{x(x-4f)}{4}}.$$

If $x$ be less than $4f$, $u$ is imaginary; in other words the least value of $x$ or $Qq$ is $4f$.

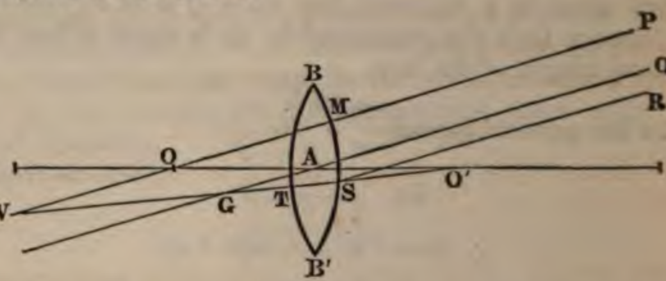
Hence if the image of a luminous point $Q$ be received upon a screen, and the lens and screen moved until the distance of the image from $q$ is the least possible, the focal length of the lens is one fourth of this distance.

If the lens be concave, it must be placed in contact with a convex lens, the focal length of which has been determined; and the convex lens must be of such power that the focal length of the combination is negative. Then if the focal length of the combination be found by the method just explained, that of the concave lens will be known by the formula of Art. 57.

[The preceding articles contain a sufficient investigation of the properties of lenses for the purposes of this treatise; but as in the case of reflexion and refraction at a spherical surface, we shall subjoin one or two propositions, which differ from some of those already given only in the mode of enunciation and the method of demonstration.

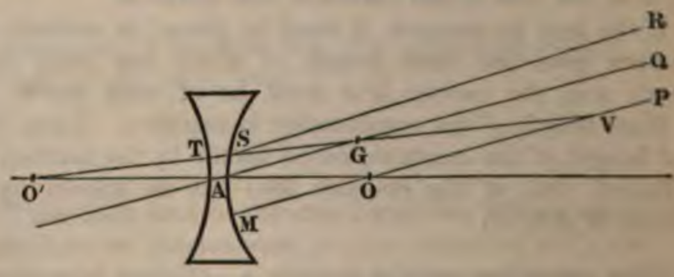
59. *To determine by geometrical construction the principal focus of a lens of inconsiderable thickness.*

We shall give the figures for two cases, the double convex and double concave lens.




Let $BAB'$ be a lens, $A$ its centre, $O$, $O'$ the centres of the first and second surface respectively. Let $QA$ be the axis of a small pencil of parallel rays, incident very nearly parallel to the axis of the lens; then the ray $QA$ passes through the lens without deviation, and therefore the focus of the refracted rays is upon $QA$ or $QA$ produced.

Through $O$ draw $OP$ parallel to $QA$, and let it meet the first surface in $M$, then the ray $PM$ of the incident pencil suffers no deviation at the first surface. In $OP$ or $OP$ produced take $MV = \mu OV$; join $VO'$ and produce it, if neces-




sary, till it cuts $QA$, or $QA$ produced, in $G$, and the first and second surface in $S$ and $T$ respectively. Draw $SR$ parallel to $QA$.

Then all the rays in the small pencil $PMSR$, will, after the first refraction, converge to or diverge from $V$; and in this

state they will fall upon the second surface; but of this pencil the ray $ST$ suffers no refraction, since its direction passes through $O'$; therefore the focus of refracted rays will be upon $ST$ produced.

But the focus of refracted rays is upon $QA$, or $QA$ produced; hence the focus of refracted rays is $G$, the point of intersection of $QA$ and $ST$ produced.

By similar triangles $OVO'$, $AGO'$, we have

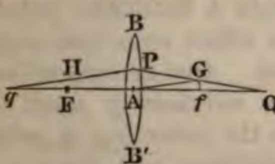
$$OO' : OV :: AO' : AG;$$

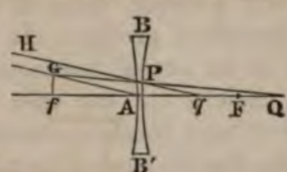
which determines the distance of the principal focus from the centre of the lens.

60. *When a small pencil of diverging or converging rays is incident directly upon a lens; the distance of the focus of incident rays from the centre of the lens is a mean proportional between the distances of the same point from the principal focus of rays coming in the opposite direction and the geometrical focus.*

In this, as in the preceding proposition, we shall give the figures for the double convex and double concave lens.

Let $BAB'$ be the lens, $A$ its centre, $Q$ the focus of incident rays. Let $QP$ be any ray incident, making a very small




angle with the axis. Let $PH$ be the emergent ray, $q$ the point in which $PH$ produced cuts the axis of the lens.

Take $f$ the principal focus of rays coming in the opposite direction; with centre $A$, and distance $Af$, describe a circular arc $fG$, meeting $QP$ or $QP$ produced in $G$. Join $GA$.

Then $G$ is the principal focus of a pencil of rays incident from the *left* (as the figures are drawn) parallel to $GA$. Consequently any ray incident from the left parallel to $GA$ will after refraction through the lens pass through $G$. And

therefore, conversely, any ray proceeding from the *right*, and passing through $G$, will after refraction be parallel to $GA$. But $QP$ proceeds from the right and passes through $G$, and $Pq$ is its direction after refraction; hence $Pq$ is parallel to $GA$. Therefore $GAQ$, $PqQ$ are similar triangles;

$$\therefore QG : QA :: QP : Qq;$$

or ultimately $$Qf : QA :: QA : Qq.$$

It is almost unnecessary to remark, that from the results of this and the preceding article, the fundamental formula of Art. 45 may be deduced. The student will find it a good exercise to deduce the one from the other.]

## ON IMAGES FORMED BY REFLEXION OR REFRACTION.

61. When light is incident from a luminous point upon a reflecting or refracting surface, or upon a combination of surfaces, whether plane or spherical, the investigations of the preceding pages enable us to determine the focus of the reflected or refracted rays; and the focus so determined is to be considered as the *image* of the luminous point, that is, the rays will proceed from it as from a luminous point, and will therefore produce the same effect upon our organs of vision as an actual luminous point. Our investigations only apply strictly to the case of an indefinitely small conical pencil of rays, incident *directly* on the reflecting or refracting surface, that is, having its axis coincident with the axis of the surface; if the incidence be *oblique*, that is, if the axis of the pencil be inclined at some angle to the axis of the surface, the reflected or refracted pencil will manifestly be altered in its form, and our formulæ will not be strictly correct. Nevertheless, since in practice the obliquity is generally small, and since the consideration of the general question of the form of oblique pencils would lead us into calculations more complicated than we desire to introduce into this treatise, we shall suppose that the formulæ, already established for direct, will

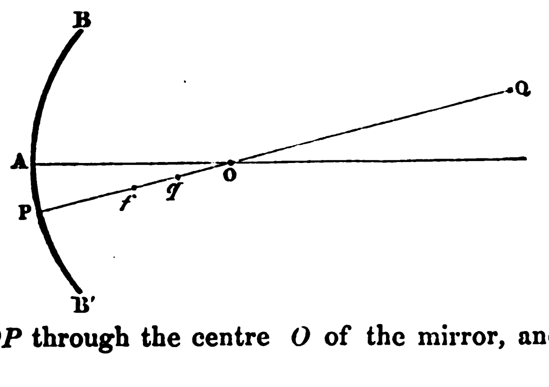
also hold for oblique incidence; a supposition which the student will bear in mind is only approximately true.

62. Having then solved the problem of finding the image of a point, we may now proceed to consider the more general one of finding the image of any object formed by reflexion or refraction. We may conceive any object to be made up of physical points, each of which is a focus of incidence, and has a corresponding focus of reflexion or refraction; if then we construct the geometrical foci, corresponding to all different points of the object, we shall have the image of the object required.

We shall confine our attention almost exclusively to the case of the image of a small straight line, because this is the simplest figure which the object can have, and the determination of the image in this case will be sufficient for our purpose, when we come to apply our results to the construction of optical instruments.

63. We shall first, however, shew how to apply our formulæ to the finding of the image of a point, from which the rays fall obliquely on a mirror or a lens.

Let $BAB'$ be a concave spherical mirror, and $AO$ its axis. Let $Q$ be a luminous point not on the axis of the mirror.




Draw $QOP$ through the centre $O$ of the mirror, and take $q$ such that

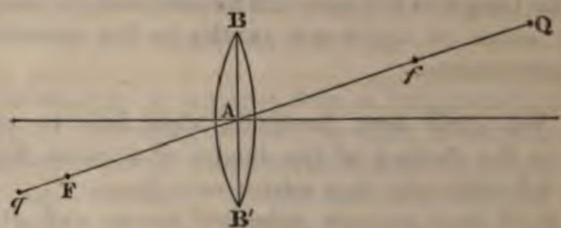
$$\frac{1}{Pq} + \frac{1}{PQ} = \frac{2}{PO};$$

then $q$ will be the image of $Q$.

The position of $q$ may be determined very easily and sufficiently nearly for many purposes, by means of the investigation of Art. 22. Bisect $OP$ in $f$, then if $Q$ were at an infinite distance $q$ would be at $f$; but as $Q$ moves towards $O$, $q$ also moves towards $O$, therefore when $Q$ has the position given to it in the figure, $q$ is somewhere between $f$ and $O$. In like manner, if $Q$ were between $O$ and $f$, $q$ would be on $PO$ produced; and if $Q$ were between $f$ and $P$, $q$ would be on $OP$ produced.

In like manner we may find the image of a luminous point in the case of a convex mirror.

64. Next let us consider a lens. Let $BAB'$ be a double convex lens; $Q$ a luminous point not in its axis. Let $A$ be




the centre of the lens, which, on account of the thinness of the lens, we may suppose to be any point of that portion of its axis which lies within the lens; for distinctness' sake, we will suppose the centre to be the point, in which the line joining $B$ and $B'$ cuts the axis. Draw $QAq$ through the centre of the lens; then, by the property of the centre, this ray will undergo no deviation, and consequently the image of $Q$ must be somewhere on the line $QAq$. If we suppose the formula proved in Art. 45, to apply to this case of oblique incidence, the distance $(Aq)$ of the image from $A$ will be given by the formula,

$$\frac{1}{Aq} - \frac{1}{AQ} = -\frac{1}{f}.$$

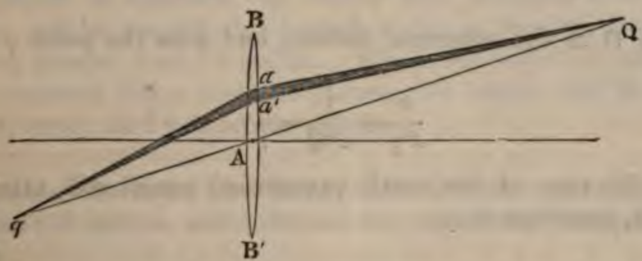
But, as in the case of reflexion, we may determine the position of $q$ sufficiently nearly for many purposes, by consi-

dering that if $AQ=f$, (the focal length of the lens,) $q$ is at an infinite distance from $A$, and that as $Q$ moves away from $A$, $q$ moves from the left towards $A$, and will have some position to the left of $F$.

In like manner, we may determine the image of a luminous point, in the case of a double concave or any other lens.

65. The same method may be adapted to the case, in which a small pencil of rays falls *excentrically* on a lens, that is, in which the axis of the incident pencil does not pass through the centre of the lens. For we may conceive a complete pencil, having its axis passing through the centre, to fall upon the lens, and we may determine the image in this case, and the same will be the position of the image when the pencil is so restricted that it becomes excentrical; the difference will be, that there will be fewer rays diverging from, or converging to, the geometrical focus.

Suppose, for instance, a very small excentrical pencil falls from $Q$ on the portion $aa'$ of a double convex lens $BAB'$:




draw $QAq$ through the centre of the lens, and take $q$ such that

$$\frac{1}{Aq} - \frac{1}{AQ} = -\frac{1}{f};$$

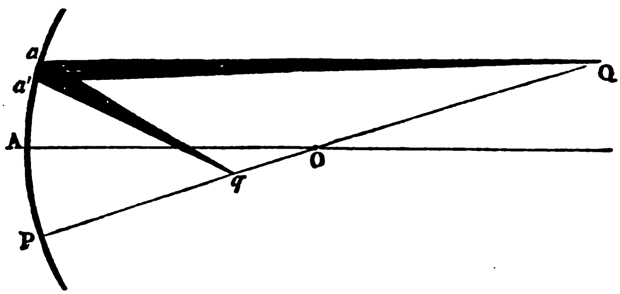
then the rays incident from $Q$ will, after refraction through the lens, converge to $q$.

If $AQ=f$, $Aq=\infty$, and the refracted rays are parallel; hence, when a small pencil of rays is incident on a convex lens from a point at a distance from it equal to its focal length, we must draw a line through the point and the centre

of the lens, and the rays will emerge parallel to that straight line.

66. A similar method is applicable to a small excentrical pencil, incident on a mirror. A pencil is in this case said to be excentrical, when the point in which its axis meets the mirror is not that in which the axis of the mirror* meets it.

Let $Q$ be the origin of a small pencil, incident on the small portion $aa'$ of the mirror. Draw $QOqP$ through the




centre $O$ of the spherical surface, and take the point $q$ such that

$$\frac{1}{Pq} + \frac{1}{PQ} = \frac{2}{r};$$

then the rays of the small excentrical pencil will, after reflexion, converge to $q$.

67. We are now prepared to consider the formation of the image of a straight line, placed before a reflecting surface, or before a lens.

The image of an object, placed before a plane reflector, will manifestly be precisely similar to the object, and each point of the image will be as far behind the mirror, as the point of which it is the image is distant from the mirror. This follows at once from Art. 12.

* By the *axis of the mirror* is to be understood here the line drawn from the centre of the spherical surface to the central point of the mirror; the mirror itself being a portion of a sphere cut off by a plane.

When a small straight line is placed before a spherical mirror, its image, that is, the locus of the geometrical foci corresponding to the different points of it, will evidently not be a straight line. If the line be indefinitely distant from the mirror, the image will evidently be a small arc of a circle; in other cases it may be shewn to be a portion of a conic section, but this we shall not do, since we shall not be concerned with the particular form which the image assumes: if however the object be supposed to be extremely small, it will be sufficient to consider its image to be also a straight line, and the only point with which we shall engage ourselves will be the determination of the position of this straight line.

68. We shall here give results, which the student will have no difficulty in verifying for himself.

*Concave Mirror.* When the object is at a distance from the mirror greater than its radius, the image will be small, inverted, and between the centre and principal focus. When the object is between the centre and principal focus, the image will be magnified, inverted, and at a distance from the mirror greater than its radius. When the object is between the principal focus and the mirror, the image will be magnified, erect, and behind the mirror.

*Convex Mirror.* The image will always be small, erect, behind the mirror, and between the mirror and its principal focus.

*Concave Lens.* The image will be small, erect, on the same side of the lens as the object, and between the lens and its principal focus.

*Convex Lens.* When the object is at a distance from the lens greater than its focal length, the image is on the opposite side of the lens, inverted, and at a distance from the lens greater than its focal length. When the object is between the lens and its principal focus, the image is erect, and on the same side of the lens as the object.

## ON THE EYE.

69. The theory of the formation of images, which we have been explaining, is applicable to the explanation of the construction of the eye; which, to take the simplest view, may be conceived of as a convex lens, by means of which images of external objects are formed upon a screen behind it, these images affecting the brain by means of nerves, and in some inexplicable manner conveying to the mind the sense of vision*.


70. The figure represents a horizontal section of the human eye; its general form is spherical, but the front $BAB'$ is more convex than the rest.

The outer coat is called the *sclerotica*, and is white and opaque, except in the front, which is occupied by the transparent convex portion $BAB'$ called the *cornea*.

The interior of the *sclerotica* is lined with a soft thin coat, called the *choroid membrane;* at the junction of this membrane with the sclerotica arises the *uvea*, an opaque membrane, having an aperture $aa'$ in its centre called the *pupil,* through which light enters the eye, and by the spontaneous enlargement and diminution of which the quantity of light admitted is regulated. The diameter of the pupil varies from about $\frac{1}{10}$th to $\frac{1}{4}$th of an inch.

The interior of the choroid membrane is covered with a black substance called the *pigmentum nigrum*, the office of which is to absorb stray rays of light, and so prevent internal reflexions, which would be the source of much confusion. At the back of the eye, and imbedded in the pigmentum

* In Dr Young's Lectures on Natural Philosophy there is a Lecture on Vision (Lecture XXXVIII) which may be studied with advantage. The subject is of necessity treated very briefly in the text.

nigrum, is the *retina*; the retina is a network of very fine nerves, which branch off from, and may be looked upon as a continuation of, the optic nerve, which proceeding directly from the brain enters at $O$, on the side of the eye next the nose.

$CC'$ is a soft, transparent substance, in the form of a double convex lens, and called the *crystalline lens.*

The space between the crystalline lens and the cornea is filled with a transparent fluid resembling water, and called the *aqueous humour*. That between the crystalline and the retina is filled with another humour, called the *vitreous humour*. The refractive indices of these humours differ very little from that of water.

71. When a pencil of light diverges from a luminous point upon the exterior surface of the eye, it suffers refraction at the cornea, and again at the surface of each successive humour through which it passes, and by the combined refraction of all is made to converge to a point upon, or very near to, the retina. In like manner the image of an external object is formed upon the retina, each point in the object having its corresponding point in the image.

It is a remarkable fact that there is a portion of the retina which is insensible to the impression of rays of light. It is that spot at which the optic nerve enters the eye, and at which the nerve is not yet subdivided into the fibres which constitute the retina. This spot is called the *punctum cæcum*, and its existence may easily be detected as follows. Place two patches of white paper upon a dark wall, the line joining them being horizontal and about the height of the eye from the ground. If then one eye be closed and the other directed to one of the patches, (the one to the left hand if the right eye be used, and *vice versâ*,) the other, to which the eye is not directed, becomes invisible on retiring from the wall to about four or five times the distance of the patches from one another, and the distance being further increased, becomes again visible. The experiment is made more remarkable by placing

a third patch beyond this in the same right line, which will continue visible when the middle one disappears*.

72. The focal length of the eye is not constant, but is varied instinctively by the eye, so as to adapt itself to vision at different distances. The shortest distance to which the eye can adjust itself varies in different persons, and is called the *least distance of distinct vision*, and this varies in different eyes from about 6 to 8 inches; with regard to vision of distant objects, the majority of persons can see when the object is at such a distance that the rays from it may be considered to be parallel to each other, but some eyes require the rays entering them to have a certain degree of divergency: in general, however, we say that rays of light are fitted to produce distinct vision, when they are parallel to each other. It may be observed, that no eye can see by means of rays having the smallest amount of convergency. The question, by what means does the eye adapt itself to different distances, has been much discussed, and apparently without any certain result; three opinions have been advocated; the first, that the effect is produced by a change of position of the crystalline lens; the second, that it is due to a change of the curvature of its surfaces, or of that of the cornea; the last, that it is produced by a change in the configuration of the whole eye and an elongation of its axis. The most plausible opinion seems to be that which combines a change of curvature of the cornea with a change in the figure of the entire eye†.

73. The eyes of animals, though agreeing in principle, are different in many of their details from those of men; the difference being, in general, such as can be accounted for, by consideration of the different circumstances of vision, to which it is desirable that they should be adapted; but the student

* This method of detecting the *punctum cæcum* is taken from Lloyd's Treatise on Light and Vision. Another method is as follows: make two blots upon a sheet of paper, about four inches apart, and look with the right eye on that which lies to the left hand, the eye being placed precisely over it. When the eye is raised to the height of about 11 inches, the second spot disappears; on continuing to lift the head, the spot will reappear when the eye is about 15 inches from the paper.

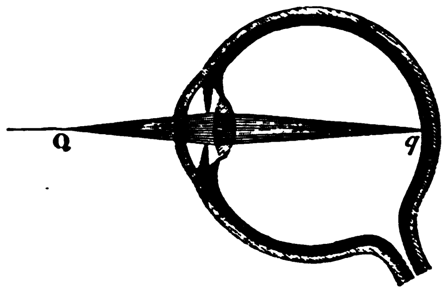
† On this subject see Lloyd's Treatise on Light and Vision.

who desires information on this head, must consult other works on the subject*.

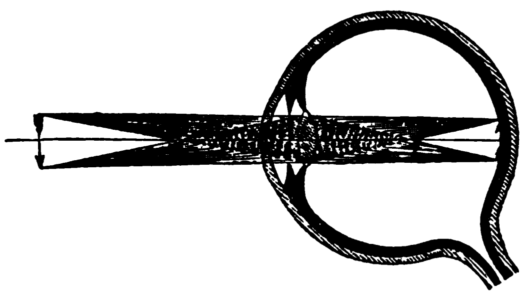
74. We shall now explain and illustrate with figures a few cases of actual vision; and for distinctness' sake we shall first consider the object viewed as a point, and secondly as of finite dimensions.

75. When rays diverge from a luminous point they diverge in all directions; and if an eye be exposed to the rays it will select a small pencil of them, limited by the magnitude of the aperture of the pupil, and provided the object be not within the least distance of distinct vision the small pencil will be made to converge to a focus on the retina.

This will be fully explained by the figure.




76. Again, suppose the object viewed to be not a point



but a small object; then what was true of the single point will

* For instance, Lloyd's Treatise before referred to.

be true of each physical point in the object, and an image will be formed on the retina as in the figure.

77. It will be observed from the last figure that the image upon the retina is necessarily inverted with respect to the object, and hence some persons have perplexed themselves by inquiring why objects do not appear to us inverted. The apparent difficulty suggested seems to rest entirely upon the confusion of two things which are very different, namely, the formation of an image on the retina, and the perception of an external object by the mind through the medium of that image. Until some light is thrown upon the mysterious agency by which the mechanical impression upon the retina is transmuted into the sense of sight, it would appear to be idle to inquire into the connexion between an inverted image on the retina, and the impression of an erect object on the mind; in fact it must be demonstrated that there is a difficulty, before we can be called upon to explain it*.

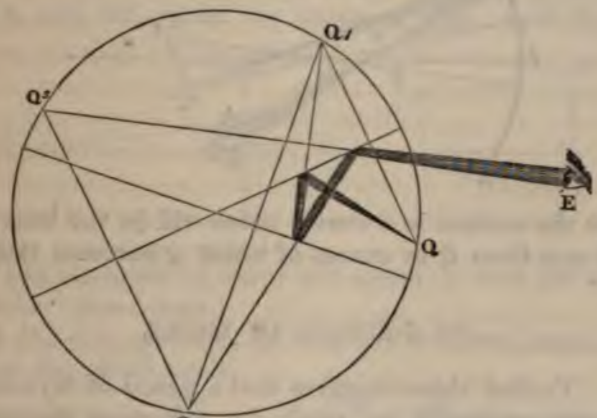
78. Next, let us consider the case of vision after reflexion at a plane surface.


Let $Q$ be a luminous point, $q$ the focus of reflected rays, $E$ an eye in any position. Then $Q$ will be seen by means of a small cone of rays having $Q$ for its vertex, and the aperture of the eye for its base; and the actual course of the pencil from $Q$ will be as in the figure.

79. If instead of a single luminous point we were to take an object of finite dimensions, then each point of the object would be seen by means of a pencil, the course of which would be found as in the preceding case.

* I do not know whether it will serve any practical purpose to say here, See Berkeley's "Essay towards a new Theory of Vision;" but certainly the student would gain great advantage from studying that Essay.

80. As another example let us take the case of a luminous point between two mirrors, as in Art. 29, and trace the pencil by means of which the luminous point will be seen after any number of reflexions by an eye situated anywhere in the plane passing through the point and perpendicular to the




intersection of the mirrors. For example, let us trace the pencil after three reflexions.

Let $Q$ be the luminous point, $Q_1$, $Q_2$, $Q_3$, the three images found as in Art. 29; $E$ the eye. Draw a line from $E$ to $Q_3$; from the point in which this line meets one of the mirrors draw a line to $Q_2$; from the point in which this line meets the other mirror draw a line to $Q_1$; and from the point in which this line meets the first mirror draw a line to $Q$. We shall thus have obtained the course of a ray from $Q$ to the eye, and the pencil can be drawn as in the figure.

81. As a last example we will consider the manner in which a small object $PQ$ is made visible to an eye $E$, (not very distant from the axis,) after reflexion at a spherical surface $BAB'$.

Let $O$ be the centre of the surface. Join $PO$, $QO$, and upon them (produced if necessary) take $p$, $q$ the geometrical foci corresponding to $P$ and $Q$. Then $pq$ will be the image of $PQ$. To determine the pencil by means of which any

point as $Q$ becomes visible, describe a cone having $q$ for its vertex and the aperture of the eye for its base; the cone will


intersect the surface in a curve, which will be the base of the cone of rays from $Q$ by means of which $Q$ becomes visible.

## ON DEFECTS OF SIGHT.

82. Perfect vision requires that a pencil of rays incident on the cornea should be made, by refraction through the several humours of the eye, to converge accurately to a point upon the retina. Vision will therefore be imperfect, when the rays converge to a point in front of the retina and then diverge upon it, or when they converge to a virtual focus behind it. The resulting defect is nearly the same in the two cases; for in both, the image of a point, instead of being a point, is a small circle or disc of light, and the image of an object is therefore formed of such small circles, which, overlapping each other, produce indistinctness or confusion.

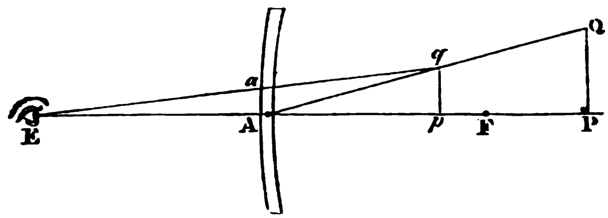
The former defect is that of *shortsight*, and arises from the too great convexity of the refracting surfaces of the eye. It may be remedied by the use of a *concave* lens, which will give the rays the degree of divergency necessary to enable the eye to bring them to a focus upon the retina.

The latter defect is that of *longsight*, and arises from the too great flatness of the refracting surfaces of the eye; it is a defect usually brought on by old age. It may be remedied by the use of a *convex* lens, which will cause the rays to enter the eye in a state of parallelism, and so to be fit to produce distinct vision.

## ON VISION THROUGH A SINGLE LENS.

83. *To determine the angle under which a given object will be seen by the eye, when viewed through a concave lens.*

Let $A$ be the centre of the lens, $E$ the eye, $PQ$ the object; join $QA$, and let $pq$ be the image of $PQ$. Join $Eq$; then




$qEp$ is the angle under which the object is seen through the lens, or the *visual angle*.

Let $PQ = \lambda$, $AP = u$, $AE = d$, $f =$ the focal length of the lens, $\theta =$ the visual angle.

Then $\tan\theta = \dfrac{pq}{AE + Ap} = \dfrac{PQ}{AE + Ap}\dfrac{Ap}{AP}$, (by similar triangles, $Apq$, $APQ$,)

$$= \frac{\lambda}{u}\frac{1}{1 + \dfrac{d}{Ap}} = \frac{\lambda}{u}\frac{1}{1 + d\left(\dfrac{1}{u} + \dfrac{1}{f}\right)}. \quad \text{(Art. 45.)}$$

Let $\alpha$ be the angle under which the object would have been seen by the naked eye, then $\tan\alpha = \dfrac{\lambda}{u + d}$;

$$\therefore \frac{\tan\theta}{\tan\alpha} = \frac{u + d}{u + d + \dfrac{du}{f}}.$$

This ratio is less than unity, and the effect is that the object appears *diminished*.

If we had taken the case of a convex lens, we should have found, that

$$\frac{\tan\theta}{\tan\alpha} = \frac{u + d}{u + d - \dfrac{du}{f}}.$$

If $u + d$ be greater than $\frac{du}{f}$, the object will appear *magnified*, and it may be magnified to any extent. But if $\frac{du}{f}$ be greater than $u + d$, $\tan\theta$ will be negative, and the image will be inverted and not necessarily magnified.

84. If we suppose the lens so adjusted that the rays enter the eye in a state of parallelism, we must have $u = f$, and then (for the convex lens),

$$\frac{\tan\theta}{\tan\alpha} = \frac{f+d}{f+d-d} = 1 + \frac{d}{f}.$$

The magnification by a convex lens will be therefore greater as $\frac{1}{f}$ is greater; hence it has been proposed, to call the quantity $\frac{1}{f}$ the *power* of a lens.

85. A very small convex lens, of short focal length, or a very small sphere of glass, may be used as a magnifying glass in a way slightly differing from the preceding. When an object is placed very near the eye, a magnified image is formed on the retina, but on account of the too great divergency of the rays the eye is not able to obtain a distinct perception of the object. If now a very small lens, not


exceeding in breadth that of the pupil of the eye, and of focal length so short that the object shall be in its principal focus, be placed close to the eye, the rays of light emerging from the lens will be parallel to each other, and therefore fit to produce distinct vision, and the magnification of the image

on the retina will be the same as before. The rays will in this case pass centrically through the lens.

The formation of the image on the retina in this case will be understood from the above figure.

Small spherules of glass have sometimes been used instead of lenses when high magnifying powers have been required, the construction of the former being easier than that of the latter when the curvature is considerable.

## ON THE GENERAL PRINCIPLE OF TELESCOPES.

86. When a small object at a great distance is viewed by the naked eye, there are two reasons why the vision is indistinct, namely, the smallness of the angle which the object subtends at the eye or the *visual angle*, and the small quantity of light which comes to the eye from any point of the object. The ends to be accomplished therefore by an instrument used for viewing distant objects are also two, namely, to increase the visual angle, and to increase the quantity of light which reaches the eye. The latter is accomplished by allowing the rays to fall upon a convex lens, called the object-glass, which collects from each point of the object a quantity of light, bearing to the quantity which would enter the naked eye, the ratio of the area of the object-glass to the area of the pupil of the eye; and the former by deflecting the rays through a system of lenses, the arrangement of which varies in telescopes of different constructions. In some telescopes the rays are received on a concave reflector, instead of a convex lens, but the principle is the same.

87. It may be seen, without any difficulty, that the two defects of vision, which are to be remedied, are in some measure antagonistic. For, suppose, that from a very small distant object a certain quantity of light falls on the object-glass of a telescope, then the magnifying power is greater in proportion as the light is spread over a *larger extent* of the retina; but the brightness of the illumination of the retina is greater in proportion as the light is more concentrated, or spread over a *smaller extent* of the retina; hence, when an

image is formed on the retina, we have this general relation,

$$\text{brightness of the image} \propto \frac{1}{\text{magnification produced}}\text{*}.$$

If therefore, with a given object-glass, we increase the magnifying power, we diminish the brightness of the image, and thus we have a limit put to increase of magnification. Hence, in order that we may have telescopes of great power, we must have object-glasses, or object-mirrors, of very large diameters; and in the construction of these lies the difficulty of making powerful telescopes.

88. We shall now proceed to describe the construction of various telescopes, premising that we shall describe them in their simplest form; to render them practically useful instruments, it would be necessary to introduce a variety of refinements of construction, which the elementary mode of treating the science of Optics adopted in this treatise precludes us from making intelligible to the student.

Telescopes may be divided into two classes, Refracting, and Reflecting. The former are more generally used; the only advantage possessed by the latter is that object-mirrors can be made of a larger size than object-glasses, and therefore reflecting telescopes can be made of greater power than refracting. But for the greater number of astronomical purposes this is no advantage; and moreover a large reflecting telescope is an instrument of much less utility than at first sight might appear, owing principally to these two circumstances, that the condensation and in cold weather the freezing of vapour upon it frequently quite unfits it for use, and that it is impossible to preserve the lustre of the polish for more than about two years together.

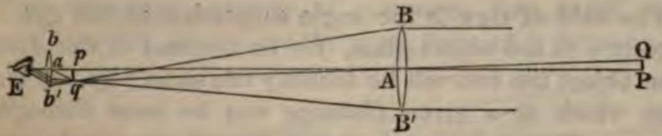
## ON REFRACTING TELESCOPES.

89. *The Common Astronomical Telescope.*

The common astronomical telescope consists of two convex lenses, a larger one $BAB'$ called the object-glass, and

* It may be observed that the magnification here spoken of varies as the square of the *magnifying power* of the telescope.

a smaller one $bab'$ called the eye-glass, placed on the same




axis, and at a distance from each other equal to the sum of their focal lengths.

Let the axis of the instrument be directed to a point $P$ of a very small object $PQ$, so distant that rays from any point of it which fall upon the object-glass are sensibly parallel; then an inverted image $pq$ will be formed in the focus of the object-glass, and the rays which diverge from any point $q$ of the image upon the eye-glass will, after refraction, emerge approximately parallel to the line $qa$, which joins $q$ with the centre of the eye-glass, since $aq$ nearly $= ap$ = the focal length of the eye-glass. If therefore an eye be placed at $E$, the point at which the axis of the pencil from $q$ crosses the axis, the rays entering the eye will be fit for producing distinct vision, and an image of $PQ$ will be seen.

Objects seen through this telescope will appear inverted, but this is of no importance in the case of celestial objects.

90. *The Magnifying Power.*

The magnifying power is measured by the ratio of the visual angles, when the object is viewed through the telescope and with the naked eye respectively.

The angle under which $PQ$ would be seen with the naked eye is $PAQ$, which $= pAq$; and the angle under which $pq$ is seen is $paq$, since the rays emerge parallel to $qa$;

$$\therefore \text{ magnifying power} = \frac{paq}{pAq} = \frac{Ap}{ap} \text{ nearly,}$$

$$= \frac{f_o}{f_e};$$

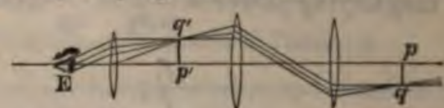
where $f_o$, $f_e$ represent the focal lengths of the object-glass and eye-glass respectively.

91. *The Field of View.*

The field of view is the angle subtended at the eye, or at the centre of the object-glass, (for on account of the distance of the object the two will be sensibly the same,) by the largest object which at a given distance can be seen through the telescope. This definition, however, though apparently precise, is not so really; for, suppose we find a point in the object from which a pencil of rays, after being refracted through the object-glass, just fall on the eye-glass: then if we take a point a little further from the axis, the rays from it will not all fall on the eye-glass, but some of them will be lost; still more will this be the case with a point a little further from the axis; and so on, until at last we come to a point from which no rays fall upon the eye-glass, and therefore none reach the eye. The result is, that in looking through a telescope such as we have described, the outer portions of the field instead of being clearly defined would gradually fade away; this imperfect part of the field is called *the ragged edge of the field of view.*

The ragged edge may be cured by placing a *stop*, or annulus of metal, in the focus of the object-glass; for by this means we can stop entire pencils of rays, which cannot be effected by a stop placed in any other position, and we can thus limit the field to any extent. If this be done, the angular extent of the field of view will be the angle subtended by the aperture of the stop at the centre of the object-glass.

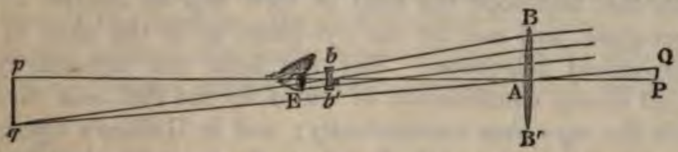
92. The telescope which we have now described is not applicable to vision of terrestrial objects, on account of its inverting property; but it may be adapted to the purpose by using a combination of lenses called *an erecting eye-piece*, instead of the simple eye-glass. The construction of such an




eye-piece will be sufficiently understood from inspection of the annexed figure, which represents one form of its construction.

93. *Galileo's Telescope.*

This telescope consists of a convex lens $BAB'$, and a concave lens $bab'$, placed on the same axis, at a distance from




each other equal to the difference of their focal lengths; $BAB'$ is the object-glass, and is of much greater breadth and focal length than the eye-glass $bab'$, which need not be larger than the pupil of the eye.

Let the axis of the instrument be directed to a point $P$ in an object $PQ$, which is at such a distance that rays from any point of it may be considered to fall upon the object-glass in a state of parallelism; then if nothing were interposed, an inverted image $pq$ would be formed of $PQ$ in the focus of the object-glass. If now an eye were placed at $E$ the rays converging to any point $q$ would not produce distinct vision (see Art. 66.); but if a small concave lens $bab'$ be placed before the eye, and at a distance from the image equal to its focal length, the rays will emerge in a state of parallelism, and therefore will produce distinct vision; and the visual angle will be $paq$, since the rays which before refraction at the eye-glass were converging to $q$ emerge parallel to $aq$. The rays being intercepted by the eye-glass before they have crossed the axis, objects will appear erect.

94. *The Magnifying Power.*

Let $f_o$, $f_e$ be the focal lengths of the object-glass and eye-glass respectively; then the visual angle for $PQ$ seen with the naked eye is $PAQ$, the visual angle when seen through the telescope is $paq$;

$$\therefore \text{ magnifying power} = \frac{paq}{PAQ} = \frac{paq}{pAq} = \frac{f_o}{f_e} \text{ nearly.}$$

95. *The Field of View.*

The ragged edge in Galileo's telescope is not curable by the use of a stop as in the astronomical telescope, because no

real image is allowed to be formed by the object-glass, and it is manifest therefore that a stop placed any where within the telescope will not stop imperfect pencils only.

In this telescope the field of view will be limited by the object-glass, and not by the eye-glass, as in the case of the astronomical telescope. For the field will necessarily be limited in any combination of lenses, by the first lens through which the rays pass excentrically; and in Galileo's telescope, the rays pass excentrically through the object-glass; for although rays fall upon the whole extent of the object-glass from each point of the object, yet the eye-glass selects a small pencil, (as will be seen distinctly from the figure,) and this small pencil, with which alone we are concerned, passes through the object-glass excentrically.

96. Galileo's telescope possesses great historical interest, as being the first combination of lenses so used; the construction is, however, now only applied to opera-glasses, and for astronomical purposes is wholly useless. The capital defect of the telescope is that no image is formed by the object-glass; now observations of the stars are made by means of fine wires, which, being placed in the focus of the object-glass of an astronomical telescope, become visible by stopping pencils of rays which there converge to points, and the place of a star is noted by referring it to these wires; but in Galileo's telescope wires cannot be so used, there being no position in which they can be made visible; hence for astronomical purposes the construction is totally useless.

## ON REFLECTING TELESCOPES.

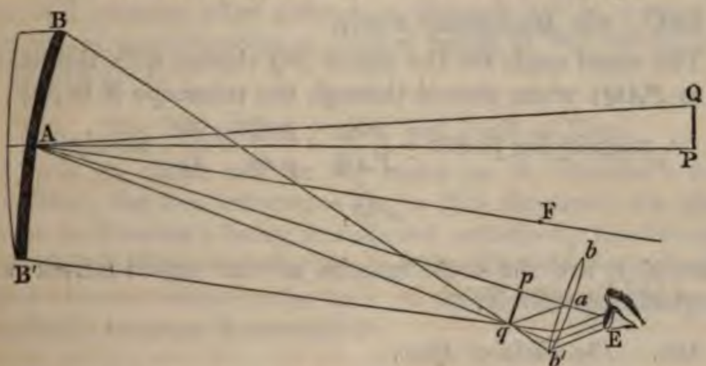
97. In reflecting telescopes, the place of the object-glass is supplied by an object-mirror, or concave speculum of metal, which reflects the incident rays and causes them to converge. We shall describe four kinds of reflecting telescopes, which involve however, (as will be seen) only two essentially different constructions.

98. *Herschel's Telescope.*

$BAB'$ is a concave speculum, having its axis $AF$ inclined

at a small angle to $AP$ the axis of the tube of the telescope, so that a ray incident in the direction of the axis of the tube is reflected in the direction $Ap$; on the line $Ap$ as axis is placed the convex lens $bab'$, at a distance from $A$ equal to the sum of the focal lengths of the mirror and lens.

If the axis of the instrument be directed to $P$, a point in a small distant object $PQ$, an inverted image $pq$ will be




formed on $Ap$, and at a distance from $A$ equal to the focal length of the mirror: and this image being by the construction in the focus of the eye-glass, the rays after refraction through it will emerge in a state of parallelism, and will therefore be fit for the production of distinct vision. Objects will appear inverted.

The form of the object-mirror should be parabolical, not spherical. See Art. 25.

99. This is the most simple construction of the reflecting telescope, and is nearly analogous to that of the astronomical telescope. It has a considerable advantage over the other constructions, which will be presently described, in the small number of reflexions which the rays undergo, and hence it is well adapted for viewing very faint objects, when no unnecessary loss of light can be afforded. But it has this very great defect, that the pencil which forms the centre of the field of view is not incident *directly* upon the mirror, and the reflected pencil has defects of a very serious kind owing to this obliquity. Consequently Herschel's telescope is not advantageous

when great distinctness of definition is the point principally desired. The construction is adapted only for very large instruments, because if the telescope be not large, the observer's head, when looking into the telescope, will materially interfere with the incidence of the rays: this defect is obviated by Newton's construction, which we shall presently describe.

100. *The Magnifying Power.*

The visual angle for the object $PQ$ viewed with the naked eye is $PAQ$; when viewed through the telescope it is $paq$;

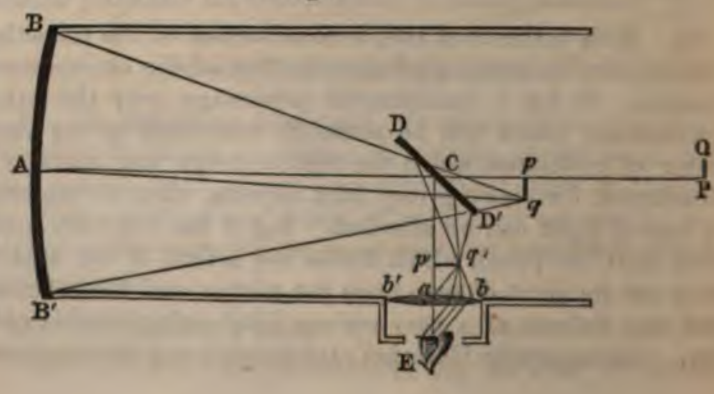
$$\therefore \text{magnifying power} = \frac{paq}{PAQ} = \frac{paq}{pAq} = \frac{ap}{Ap}, \text{ nearly,}$$
$$= \frac{f_o}{f_e},$$

where $f_o$ $f_e$ are the focal lengths of the object-mirror and eye-glass respectively.

101. *The Field of View.*

The ragged edge may be cured, as in the astronomical telescope, by placing a stop at the common focus of the mirror and eye-glass, and the field of view will then be measured by the angle which the diameter of the stop subtends at the central point $A$ of the mirror. If there be no stop, we may take the angle subtended by the eye-glass at the same point as the measure of the field of view.

102. *Newton's Telescope.*



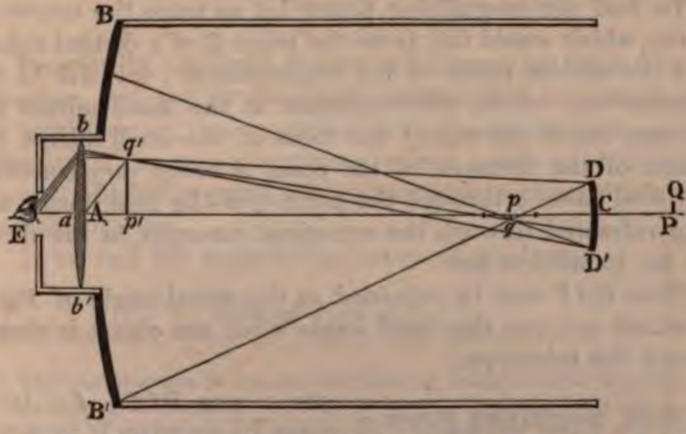
*BAB'* is a concave mirror, which, if light were incident upon it from a small distant object *PQ*, would form an inverted image *pq* of *PQ* in the principal focus. But a small plane mirror *DCD'*, placed at an angle of 45° with the axis of the object-mirror, causes the image to be formed at *p'q'*, instead of *pq*, and if a convex lens *bab'* be placed on *Cp'* as axis, and at a distance from *p'q'* equal to its focal length, the rays will emerge after refraction through *bab'* in a state of parallelism, and therefore a distinct image of *PQ* will be seen by an eye at *E*.

103. *The Magnifying Power, and Field of View.*

Both of these will be the same as in Herschel's construction; the two telescopes are in fact the same, the plane mirror in Newton's being introduced principally for the sake of avoiding the necessity of looking directly towards the object-mirror, which in the case of small instruments would manifestly be most inconvenient.

104. *Gregory's Telescope.*

*BAB'* is a concave mirror, which being directed to a distant object *PQ* forms an inverted image of it *pq* in its principal

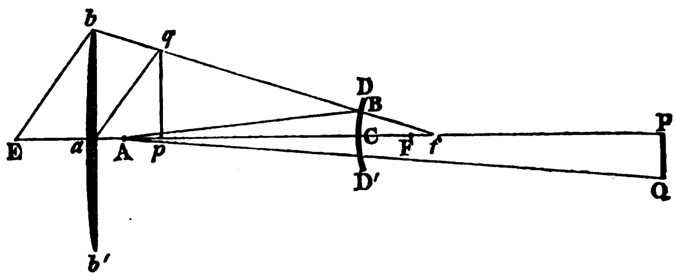



focus. *DCD'* is a small concave mirror, on the same axis as *BAB'*, and at a distance from *p* rather greater than its focal length; the rays from the different points of *pq* diverge upon

$DCD'$, and after reflexion form an inverted image $p'q'$ of $pq$, which will therefore be erect with respect to $PQ$. The adjustment is such that $p'q'$ is formed in the focus of a convex lens $bab'$, on the same axis as the two mirrors, and on which the light falls through a circular aperture in the middle of the object-mirror; the rays of light therefore, after refraction through it, emerge in a state of parallelism, and produce distinct vision of the object $PQ$ to an eye at $E$.

Objects seen through this telescope will appear erect.

105. *The Magnifying Power.*




To find the magnifying power let us trace the course of the ray, which *would* fall, from the point $Q$ of a distant object, on $A$ the middle point of the object-mirror; let $AB$ be the reflected ray, which, after reflexion at the small mirror will pass very nearly through $f$ the focus of the small mirror, (on account of the distance of the point $A$ being very considerable,) and passing through the point $q$ of the final image, and being refracted through the eye-glass, emerges in the direction $bE$, parallel to $qa$.

Then $QAP$ may be regarded as the visual angle of $PQ$ to the naked eye, $qap$ the visual angle when the object is viewed through the telescope;

$$\text{therefore, magnifying power} = \frac{qap}{QAP} = \frac{qap}{qfp}\,\frac{BfC}{BAC} = \frac{fp}{ap}\,\frac{AC}{fC}$$

$$= \frac{f_o^2}{f_e f_m} \text{ nearly,}$$

where $f_o f_m f_e$ are the focal lengths of the object-mirror, small mirror, and eye-glass respectively*.

106. *Field of View.*

The field of view may be limited either by the eye-glass, or by the small mirror. With the same figure as in the last article, suppose the ray there traced to fall on the extreme point $b$ of the eye-glass, then the eye-glass will limit the field, and we shall have,

the field of view $= 2PAQ = BAC$

$$= 2\frac{fC}{AC} \cdot BfC = 2\frac{fC}{AC} \cdot afb = \frac{f_m}{f_o^2}\, b \text{ nearly},$$

where $b$ is the breadth of the eye-glass.

If the field of view is limited by the small mirror, we must suppose the ray traced in the figure to pass through the extreme point $D$ of the small mirror, in which case the field of view $= \dfrac{DD'}{AC} = \dfrac{c}{f_o}$, where $c$ is the breadth of the small mirror.

In order that the field, as limited by these two considerations, may be the same, we must have

$$\frac{c}{b} = \frac{f_m}{f_o}.$$

Practically, the field of view will always be limited by the eye-glass, and not by the small mirror; if it were limited by the small mirror, the construction of the instrument would be defective. The preceding investigation points out the smallest breadth which the small mirror can have without diminishing the field of view.

If we call the magnifying power $M$, we have

$$\text{field of view} = \frac{b}{f_e} \cdot \frac{1}{M}.$$

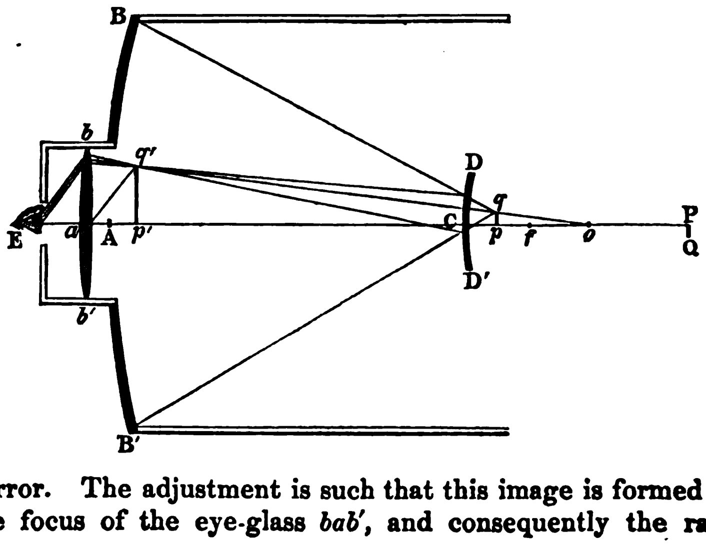
* The approximations in this and the following article are of a very rough kind, $AC$ and $fp$ being each considered to be equal to $f_o$. But it will be easily seen that little value would be attributable to a more careful approximation, since a rough estimate of the power of the telescope is all which is required in practice. It may be observed, however, that the expression given in the text is a more accurate value of the ratio of the angles $gap$, $QAP$, than it would seem to be from the manner in which it is obtained.

From this expression it appears that a telescope of high magnifying power will have a very small field of view, in consequence of which it is very difficult to bring such an instrument to bear upon a heavenly body. Hence powerful telescopes are always supplied with a small telescope, called a *finder*, having its axis parallel to that of the larger instrument; the heavenly body having been found with this less powerful telescope, and brought into the centre of its field, will then also be in the centre of the field of the more powerful instrument.

107. *Cassegrain's Telescope.*

The construction of this instrument differs from that of the preceding, only in having a convex small mirror instead of a concave.

$BAB'$ is a concave mirror, which, being directed to a distant object $PQ$, would form an inverted image of it $pq$ in its principal focus; but the reflected rays are intercepted by the small convex mirror $DCD'$, which is so placed that its focus is a little further from the object-mirror than the principal focus of that mirror, and consequently an image $p'q'$, inverted with respect to $PQ$, is formed at some distance from the small



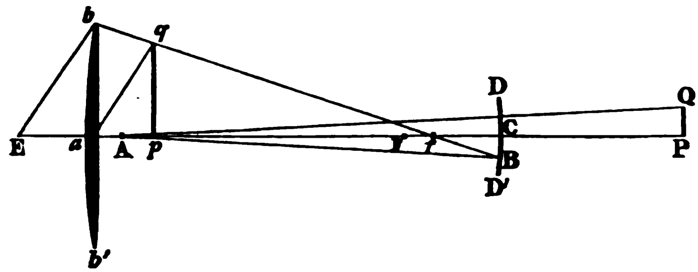

mirror. The adjustment is such that this image is formed in the focus of the eye-glass $bab'$, and consequently the rays

after refraction emerge in a state of parallelism, and are therefore fitted to produce distinct vision.

Objects seen through this telescope will appear inverted.

108. *The magnifying power.*

Let a construction be made similar to that for Gregory's telescope.




$$\text{Then the magnifying power} = \frac{qap}{QAP} = \frac{qap}{qfp}\,\frac{BfC}{BAC}$$

$$= \frac{fp}{ap}\,\frac{AC}{fC} = \frac{f_o^2}{f_e f_m} \text{ nearly.}$$

109. *Field of View.*

If the field be limited by the eye-glass we shall have,

$$\text{field of view} = 2PAQ = 2BAC$$

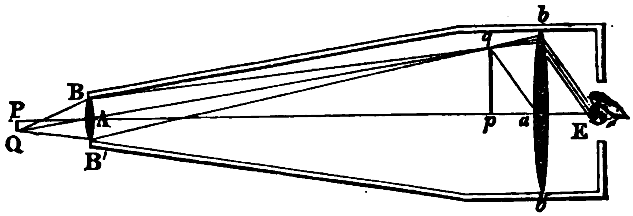
$$= 2\frac{fC}{AC}BfC$$

$$= \frac{f_m}{f_o^2}b, \text{ nearly.}$$

110. *The Compound Refracting Microscope.*

We have already explained (Art. 85.) the principle upon which a small convex lens of very short focal length may be used as a magnifying glass, or simple microscope. Combinations of lenses may be used for the same purpose, or combinations of reflectors and lenses, and such combinations are called *compound* microscopes. We shall confine ourselves to the description of the compound refracting microscope in its simplest form, observing that to make it practically useful a number of refinements must be introduced.

$BAB'$ is a small convex lens, before which, and at a distance from it a little greater than its focal length, if a small object $PQ$ be placed, an inverted image $pq$ will be




formed of it. The adjustment is such that $pq$ is formed in the focus of a convex lens $bab'$, and therefore the rays when refracted through it emerge in a state of parallelism, and therefore in a state fit to produce distinct vision; and an eye at $E$ will see a magnified inverted image of $PQ$.

The method of estimating the magnifying power of the microscope requires notice. For the magnifying power will not be measured by the ratio of the angles $qap$, $qAp$ as in the case of a telescope, but this ratio must be multiplied by the ratio which the distance of distinct vision with the naked eye bears to the distance $AP$, since the angle under which the object is seen must be compared with that under which it would be seen if viewed under ordinary circumstances with the naked eye.

Suppose $f_o$ the focal length of $BAB'$, $f_e$ that of $bab'$, and let $AP = u$. Then, without regard to sign,

$$\frac{1}{Ap} + \frac{1}{u} = \frac{1}{f_o};$$

$$\therefore \frac{1}{Ap} = \frac{u - f_o}{uf_o},$$

$$\text{and } \frac{Ap}{ap} = \frac{u}{u - f_o} \frac{f_o}{f_e}.$$

Now let $\lambda$ be the ordinary distance of distinct vision, then the magnifying power will be measured by

$$\frac{\lambda}{u - f_o} \cdot \frac{f_o}{f_e}.$$

# ASTRONOMY.

# ASTRONOMY.

1. We propose in the following treatise to give some account of the physical constitution of the universe, the motions of the heavenly bodies, and the resulting phenomena, with the mode of making observations; all which and other kindred subjects are classed under the head of Plane Astronomy: we shall not here treat of the Physical branch of Astronomy, which investigates phenomena on the principles of Mechanics, and refers them to general laws, but only of that branch which deals with facts as matters of observation. We must premise, that the subject is so vast and extensive, that the student must not expect more than the merest introduction in a treatise which is necessarily so brief as the present.

As it is of the utmost importance that the student should be perfectly familiar with the notion of a *sphere*, and of lines drawn upon it, we shall commence by presenting him with a few of the most elementary propositions and notions belonging to the doctrine of the sphere.

## ON THE SPHERE.

2. Def. A sphere is a surface every point in which is equidistant from a given point, called its *centre*.

The distance from the centre to the surface is called the *radius*, and any line passing through the centre and bounded by the surface is called a *diameter* of the sphere.

3. *Every section of a sphere made by a plane is a circle.*


Let $EDF$ be any such section of a sphere, of which the centre is $O$. Draw $OC$ from $O$ perpendicular to the cutting plane, and join $CD$, $OD$, $D$ being any point in the section $EDF$.

Then since $OC$ is perpendicular to the cutting plane, it is perpendicular to any line in that plane, and therefore to $CD$;

$$\therefore\ OD^2 = OC^2 + CD^2,$$

$$\text{or } CD^2 = OD^2 - OC^2;$$

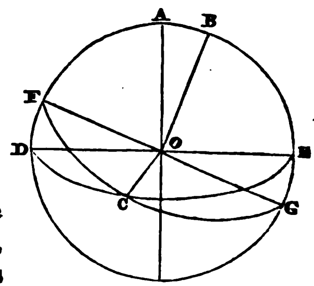
but $OD$ and $OC$ are both constant quantities, therefore $CD$ is constant, or the section is a circle having $C$ for its centre.

4. A section of a sphere made by a plane passing through the centre is called a *great* circle; other sections are called *small* circles.

The diameter of the sphere, which is perpendicular to the plane of any circle on the sphere, is called the *axis* of that circle; and the points in which the axis meets the sphere are called the *poles* of the circle.

It is evident that the poles of a great circle are equidistant from it; also it is easy to see that the pole of a circle is equidistant from every point in the circle.

5. *The angle which is subtended at the centre of a sphere by the arc joining the poles of two great circles, is the angle of inclination of the planes of the circles.*




Let $O$ be the centre of the sphere; $DCE$, $FCG$ the two great circles; $OA$, $OB$, lines respectively perpendicular to the planes of the circles, so that $A$, $B$ are their poles. Join $OC$.

Then $OC$ being in the plane $DCE$ is perpendicular to $AO$, and being in the plane $FCG$ is perpendicular to $BO$; therefore $CO$ is perpendicular to the plane in which $AO$, $BO$ lie, and therefore to $EO$ and $GO$;

$\therefore$ $EOG$ is the inclination of the planes of the circles.

$$\text{But } EOG = 90^\circ - BOE = AOB;$$

therefore the angle subtended by $AB$ is the inclination of the planes of the circles.

COR. Hence also it appears, that the arc joining the points in two great circles distant $90^0$ from the point of their intersection, subtends at the centre of the sphere an angle equal to the inclination of the planes of the circles.

6. *To determine the position of a point on a sphere.*


Let $A$ be a point on a sphere, the centre of which is $O$; then its position may be most conveniently determined as follows.

Let $POP'$ be a given diameter of the sphere; through $P$, $A$ and $P'$ draw the great circle $PAP'$; then if the angle which the plane of $PAP'$ makes with a given plane passing through $PP'$, and the arc $PA$, be given, the position of $A$ will be completely determined.

## ON THE FIGURE OF THE EARTH.

7. The form of the earth is nearly, but not accurately, spherical. Its true form is that of a slightly oblate spheroid, or a surface generated by the revolution of an ellipse, having its axes nearly equal, about its minor axis. In the greater number of cases it is sufficient to consider the earth's figure to be that of a sphere, having a radius of about 4000 miles.

The round form of the earth is easily concluded from such considerations as the following; the tops of the masts are the first portion of a ship which become visible; all the heavenly bodies with which we are acquainted have that form; moreover the earth has been actually sailed round; and lastly, in the case of a lunar eclipse the shadow of the earth may be seen upon the face of the moon and its circular form is then exhibited. The experiments which determine the actual figure to be spheroidal are of a far more delicate kind, and cannot be entered upon here; the general principles of the methods employed will be explained hereafter.

8. The heavens present to an observer on the earth's surface the appearance of a hollow sphere, at the centre of

which the observer stands; and it will be convenient to conceive of such a sphere, which we may call the *celestial sphere*, and in the surface of which we may conceive the heavenly bodies to be; the actual point on the surface of the celestial sphere, to which we shall refer any given object, will be the point in which the line joining the eye of the observer and the object meets the surface of the sphere.

On account of the enormous distance of the heavenly bodies from the earth, it will be, for many purposes, indifferent whether we consider the centre of the earth or the position of the observer as the centre of the celestial sphere.

9. The earth revolves, as will be explained more particularly presently, about a certain axis coinciding very nearly with the minor axis of its figure, considered as a spheroid; the points in which this axis produced meets the celestial sphere are called the *North and South Poles*. The great circle of which these points are the poles is called the *equator*, and the two equal portions into which the plane of the equator divides the celestial sphere are called the *Northern and Southern Hemispheres*.

The plane of the equator cuts the surface of the earth into two equal portions, which are also called respectively the northern and southern hemispheres; and the circle in which the plane cuts the earth is sometimes called the equator, as well as that in which it cuts the celestial sphere.

The direction of a line perpendicular to the surface of still water at any place on the earth's surface, is called the *vertical* at that place; and the points in which the vertical line meets the celestial sphere, are called the *zenith* and *nadir* of the place. The vertical direction will be very approximately that of the line joining the place with the earth's centre. The actual direction of the vertical may be ascertained by suspending a plumb-line, that is, a fine wire which carries a weight sufficiently heavy to keep it thoroughly stretched. The vertical direction may also be spoken of as the direction of the force of gravity, or that in which a body falls to the earth*.

* See this subject explained with great care in the "Elementary Chapters in Astronomy."

A plane perpendicular to the vertical at the earth's surface, is called the *sensible horizon;* a plane perpendicular to the same line at the earth's centre, the *rational horizon.* In the greater number of cases, the sensible and rational horizon may be considered as coincident.

The *meridian* of a place is the great circle passing through the poles of the heavens and the zenith of the place.

The intersection of the plane of the meridian with the horizon is called the *meridian line;* the intersections of this line and a line at right angles to it with the celestial sphere are the four *Cardinal points,* North, South, East, and West; looking to the North that on the right hand is the East, that on the left the West.

## ON TERRESTRIAL LATITUDE AND LONGITUDE.

10. The position of a place upon the earth's surface may be determined on the principle explained in Art. 6. Let the meridian of some place, as Greenwich, be considered as given; then the angle between the meridian of Greenwich and that of the place in question is called the *longitude* of the place, and the angle subtended by the arc of the meridian between the zenith of the place and the equator is called the *latitude;* and the latitude is said to be *north* or *south,* according as the place is to the north or south of the equator. The latitude and longitude being given, the position of the place is defined. The latitude of Greenwich is $51^{\circ} 28' 40''$ North.

The complement of the angle which measures the latitude of a place is called the *co-latitude;* it will be measured by the arc of the meridian between the zenith and the pole.

The latitude of a place may also be spoken of as the elevation of the pole of the heavens above the horizon of the place. Circles parallel to the equator are called *parallels of latitude;* the latitude of all places on the same parallel is obviously the same. The length of a degree of latitude upon the earth's surface is about 70 English statute miles.

It is usual to measure longitude through 180° east and west of Greenwich; perhaps it would be more convenient to measure through 360° in the same direction, but in practice longitude is never so reckoned.

## ON THE EARTH'S MOTION.

11. The motion of the earth may be conceived of, as being compounded of two motions, namely, a motion of revolution about an axis, while at the same time that axis is moving parallel to itself in space.

Let us first consider the revolution about the axis: and for a first approximation we may say, that the earth revolves about a line coinciding with its shorter axis and remaining fixed in space; this we shall find afterwards to be not strictly true. The time of revolution is twenty-four hours*; and the effect produced to an observer on the earth's surface is this, that, imagining himself to be fixed in position, the celestial sphere appears to revolve about its poles, carrying the heavenly bodies with it; so that the sun and stars describe circles about the axis of revolution, the greater part small circles, those only describing great circles which happen to be in the equator. When a heavenly body comes into the horizon of any given place it is said to *rise*, when it reaches the meridian it *culminates*, and when it again reaches the horizon it *sets*. The heavenly bodies appear to revolve round the earth from east to west, consequently the revolution of the earth takes place from west to east.

All this coincides with observation; for the stars are observed to revolve about a certain point in the heavens, nearly coinciding with a bright star, known as the Pole Star, and we are therefore obliged to adopt one of two hypotheses, namely, that the celestial sphere actually revolves about the earth as fixed, or that the celestial sphere being fixed the earth revolves about an axis which remains fixed in space.

* We are here using the term *hour* in a popular sense; this is the case with many words which are necessarily introduced for the purpose of explanation, and which only receive a strict definition from the refined processes of Astronomy.

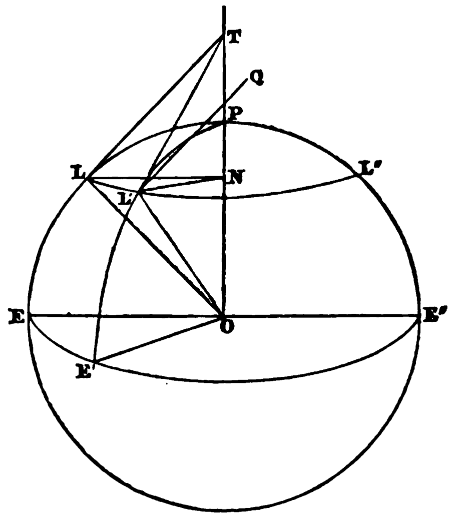
The great simplicity of the latter hypothesis leaves little doubt concerning its truth. Moreover the analogy of the other heavenly bodies leads to the same result; the sun for example is known from observation of spots upon its surface to revolve upon its axis in somewhat more than 25 days. The rotation of the earth is also connected in a remarkable manner with its form; the oblate form of the earth being a necessary consequence of the centrifugal force of the particles arising from its rotation.

The rotation of the earth has lately been demonstrated in a novel and curious manner, which deserves notice. The peculiarity of the demonstration consists in the fact of the rotation being rendered palpable by its effects upon a pendulum; if a heavy ball be suspended by a fine thread and made to oscillate, and if the earth upon which the ball is suspended be revolving about an axis it seems probable that the effect of that revolution will in some way influence the nature of the oscillation; the actual nature of the effect it is rather difficult to conceive. We will however attempt to exhibit it.

Let us first take a particular case, in which the problem presents no difficulty. Suppose the point of suspension of the thread to be upon the axis of the earth's rotation, or the pendulum to be made to oscillate at the North Pole. Then the rotation of the earth will in no way influence the direction of the pendulum's motion; and consequently if we make a mark immediately below the pendulum, shewing the direction of its vibrations at any moment, this mark will turn round while the direction of the vibrations remains the same, and to a spectator who moves with the earth the mark will be at rest, and the plane of vibration will have an *azimuthal* motion round the pole.

Now let us consider how this effect will be modified if we suppose the pendulum suspended at any other point of the earth's surface. Let $O$ be the earth's centre; $P$ the north pole, $EE'E''$ the earth's equator, $L$ the place of suspension which is carried by the supposed diurnal rotation of the earth in the circle $LL'L''$. Draw $LT$ a tangent to the circle $PLE$

which is the meridian of $L$; then if we suppose for simplicity's




sake that the pendulum is made to oscillate in the plane of the meridian, $LT$ will be the direction of the oscillations. Let $L'$ be a position of the place $L$ indefinitely near to $L$, or if we draw the circle $PL'E'$, let this be the position assumed by the meridian of $L$ after an indefinitely small amount of rotation of the earth. Draw $L'T$ a tangent to $PL'E'$ which will meet the earth's axis in the same point $T$ as $LT$; also draw $L'Q$ parallel to $LT$; $LN$, $L'N$ perpendicular to $PO$; and join $LO$, $L'O$, $E'O$.

Then ultimately $LT$, $L'T$, and $L'Q$ will be in the same plane; and the force of gravity whether at $L$ or $L'$ may be regarded as a force perpendicular to this plane; the effect of the indefinitely small amount of rotation of the earth therefore will be the same as that of shifting the plane of vibration through an indefinitely small space parallel to itself; so that when $L'$ is the place of suspension $L'Q$ is the direction of the vibration. But $L'T$ is the direction of the meridian, consequently the line of vibration instead of now lying in the meridian makes with it an angle $TL'Q$ or $LTL'$. The same thing will take place for each subsequent indefinitely small

motion of rotation of the earth, and the plane of the pendulum's motion will therefore have a continuous *azimuthal* motion, but will not make a complete revolution during 24 hours because the angle $LTL'$ is manifestly less than $LNL'$ or $EOE'$.

The exact amount of azimuthal motion is easily calculated. Let $LNL' = \phi$, $LTL' = \theta$, and $\lambda = EOL = LTN$, the latitude of the place. Then

$$LT \operatorname{chd} \theta = LN \operatorname{chd} \phi,$$

or since $\theta$ and $\phi$ are indefinitely small,

$$\theta = \phi \frac{LN}{LT} = \phi \sin \lambda.$$

And the same relation will hold for all subsequent small angles of rotation of the earth, so that if $\phi$ be any angle through which the earth has turned, and $\theta$ the corresponding angle through which the plane of the pendulum has revolved, we shall still have

$$\theta = \phi \sin \lambda:$$

and when the earth has made a complete revolution, the plane of the pendulum will have turned through $2\pi \sin \lambda$.

For the pole, $\lambda = \frac{\pi}{2}$, and $\theta = \phi$, as before explained. For the equator, $\lambda = 0$, and $\theta = 0$, or there is no change produced.

The results thus shewn to be consequent upon the rotation of the earth are conformable with experience; only it must be remembered that the phænomenon will in practice be somewhat modified by the fact of the oscillations taking place in air instead of a vacuum as here supposed. Conversely, from the observed change in the plane of vibration of a pendulum the rotation of the earth may be concluded.

A simple but singular consequence results from the rotation of the earth. The sun appears to an observer on the earth's surface to rise in the east, and after passing his meridian to set in the west; now if when the sun is on his meridian a person were to travel westward at such a rate as would

carry him round the earth in 24 hours, it is evident that he would keep the sun constantly upon his meridian; and if he travel westward but more slowly, the sun will be a longer time than 24 hours before it again comes upon his meridian, and therefore if he carry a chronometer with him the chronometer will be too fast. Suppose now he travels completely round the world, then the sun will have crossed his meridian later and later every day, until when he returns to the place from which he started it will cross 24 hours later, in other words, he will have lost a whole day in his reckoning; and if he kept a journal, that which would be entered in it as Monday (for instance) would be with the inhabitants of the place from which he started Tuesday. If he travel eastward the reverse effect will take place, or he will find himself a day in advance.

This is the *diurnal* motion of the earth, which gives rise to the succession of day and night; in addition to this there is an *annual* motion, that is, the earth is carried round the sun in a certain period, which constitutes one year. In this motion the axis of revolution moves always parallel to itself, as is shewn by the fact of its appearing always to point to the same point of the celestial sphere. The centre of the earth does in fact describe an ellipse in one plane about the centre of the sun, and this ellipse does not differ much from a circle; at present, however, we are not concerned with the actual path described by the earth, but only with the fact of its moving round the sun in the plane of a great circle, in the course of a year. According to observation, the sun appears to move in that time round the earth, but the phænomena will be exactly the same, whether the earth move round the sun, or the sun round the earth, and the consideration of the enormous magnitude of the sun as compared with the earth, combined with other reasons which will appear hereafter when we come to treat of the planets, leave no doubt as to the correctness of the hypothesis of the motion of the earth about the sun, not the sun about the earth. Properly speaking, neither the one assertion nor the other is correct, for the point about which the motion takes place is the centre of gravity of the sun and planets; as how-

ever on account of the enormous mass of the sun the centre of gravity of the whole system is not very different from its own centre, it will be in general sufficient to speak of the sun being fixed and the motion taking place relatively to the sun.

For purposes of explanation however, we shall in general speak of the sun as moving in a great circle about the earth, and this great circle we shall call the *ecliptic*.

The inclination of the plane of the equator to that of the ecliptic is an angle of about 23°28′, and is called the *obliquity of the ecliptic*.

12. The rotation of the earth upon its axis is effective in producing the phænomenon of the Trade Winds, which we shall here therefore take the opportunity of explaining.

Without assuming the knowledge of subsequent explanations, it may be taken for granted that the power of the sun upon the regions of the earth in the neighbourhood of the equator is much greater than upon those nearer to the poles. The effect of the sun's heat upon the atmosphere is to rarefy the air and cause it to expand; it is therefore displaced and raised from the surface by the influx of colder and therefore heavier air from the regions more distant from the equator; thus is caused a current of air setting from the poles towards the equator, while the rarefied air rising to the higher regions flows towards the poles forming a counter-current. Putting out of consideration then the rotation of the earth we should have a regular north wind north of the equator, and a south wind south of it; but this phænomenon is modified considerably by the circumstance of the earth's rotation; for it will be observed that the velocity of a point upon the earth's surface is greater as the point is nearer to the equator, or as the point is further from the axis of rotation; also if a particle of air be at rest with respect to the earth's surface, it is so because it is moving with the same velocity as the point of the earth's surface with which it is in contact; when a particle therefore moves towards the equator it will constantly approach points of the earth's surface which are moving more rapidly than itself, and the particle of air with respect to the earth's surface will therefore

appear to lag or to have a motion in the opposite direction. Now the motion of a point on the earth's surface is from west to east, therefore the air will have with reference to the earth's surface a motion from east to west, and this motion combined with that before explained, will constitute a permanent north easterly and south easterly wind; these are known as the Trade Winds.

## ON THE SEASONS.

13. The remarkable changes of temperature which are observed to take place in the different parts of the year, which we call Spring, Summer, Autumn, and Winter, may be simply explained by reference to the fact that the axis about which the rotation of the earth takes place remains always parallel to itself and is inclined at a fixed angle to the ecliptic. This circumstance affects the temperature in two ways; in the first place it cannot escape the most ordinary observation that in summer time the altitude attained by the sun at midday is much greater than that attained in the winter; and the lower the sun's altitude the more obliquely do its rays fall upon the earth's surface, and it is manifest that if we take a small plane of given magnitude and expose it to the sun's rays, the quantity of heat which it will receive will depend upon the angle at which it is inclined to the direction of the rays, varying from zero when the plane is held parallel to the rays to a maximum when it is held perpendicular to them. In these latitudes we never obtain this maximum effect of the sun's rays, in other words we never have the sun vertically over our heads, but the more nearly the midday altitude of the sun approaches to that extreme limit the greater is the amount of heat which we receive in a given time; and the greater midday altitude of the sun in summer than in winter may therefore be taken as one cause of the greater heat of that season.

But again, the earth receives heat from the sun during the day, and loses it by the process called radiation during the night; consequently the longer the day and the shorter the night, the less will be the loss by radiation, and the higher the temperature upon the whole. Now in summer

the days are longer than the nights, in the winter the contrary; consequently we have here a second and a very important cause of the change of temperature throughout the year.

The heat received by the earth of course depends also upon the earth's distance from the sun, and this not being quite constant there will be a variation of temperature due to this cause; it is evident however that no portion of the heat of summer in Northern Latitudes is due to this cause, from the fact that the earth is nearest to the sun almost at the period of midwinter, and therefore the effect is rather to mitigate the cold of winter than to increase the heat of summer; it may be shewn, however*, that in consequence of the angular velocity of the earth being greatest when nearest to the sun, and in fact increasing according to precisely the same law as the intensity of the heat increases, a compensation takes place, so that an equal distribution of light and heat is accorded to both hemispheres: were it not for this compensation the tendency of the cause now under consideration would be to exaggerate the difference between summer and winter in the southern Hemisphere and to equalise the temperature of the two in the Northern. This does not however prevent the effect of the direct heat of the sun's rays in Southern Latitudes being greater than in our own, the excess of the intensity of sunshine due to this cause being as much as one fifteenth of the whole.

14. The manner in which the obliquity of the earth's axis produces the change of the seasons will be seen from a

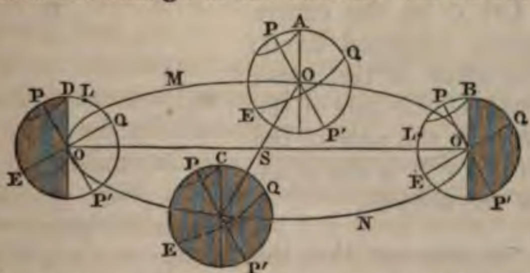



figure. Let $MN$ be the path of the earth's centre $O$ round

*See "Outlines of Astronomy," page 217.

the sun $S$. $POP'$ the axis about which the earth revolves and which is inclined at an angle of $23^{0}28'$ to a line perpendicular to the plane $MN$. $A, B, C, D$, may be supposed to represent the position of the earth on the 21st of March, the 21st of June, the 21st of September, and 21st of December respectively. The hemispheres, which are turned away from the sun, and are therefore deprived of its light and heat are shaded to denote that circumstance. Now if we consider the condition of any point in the Northern hemisphere, that is, lying between $P$ and the equator $EQ$, we shall see at once that the two conditions of heat, as depending upon the direct incidence of the sun's rays and the shortness of the night, are realised for the position $B$ of the earth and not for the position $D$: these are the two extreme cases; and it is manifest that the sun's rays fall at a smaller obliquity upon a place $L$ in North latitude for the position $B$ of the earth, than they do upon the same place for the position $D$, indeed there is one parallel of latitude in the former case upon which the rays can fall absolutely vertically; moreover if the earth be supposed to revolve uniformly upon its axis $PP'$, it is clear from inspection that the time during which the place $L$ will be immersed in shade will be much less in the former case than in the latter. In other words for the position $B$ of the earth $L$ is in midsummer, for the position $D$ it is in midwinter. The effects for the positions $A$ and $C$ will be intermediate to these two extreme cases.

## ON THE SUN'S MOTION IN THE ECLIPTIC.

15. Let $O$ be the centre of the earth, supposed fixed; and let the plane of the paper pass through $O$, the pole of the equator $P$, and the pole of the ecliptic $\Pi$; $E \Upsilon E'$ is the equator, $S \Upsilon S'$ the ecliptic.


Then we may say, that the sun describes its course in the ecliptic round $O$ uniformly in

the course of a year; this is not strictly true, on account of the orbit of the earth round the sun being not a circle, but an ellipse of small excentricity; it will however be sufficiently nearly true to serve the purpose of the present explanation. The ecliptic is conceived to be divided into twelve equal portions, each therefore consisting of $30^0$, and these portions are called the *twelve signs of the Zodiac;* they are known by the following names: Aries, Taurus, Gemini, Cancer, Leo, Virgo, Libra, Scorpio, Sagittarius, Capricornus, Aquarius, Pisces; the origin of these names will be seen hereafter; they are denoted by different symbols, one only of which we shall use, (♈), which is the symbol for *Aries.* The first point of Aries is determined by the intersection of the equator and ecliptic, the other point of intersection being the first point of Libra.

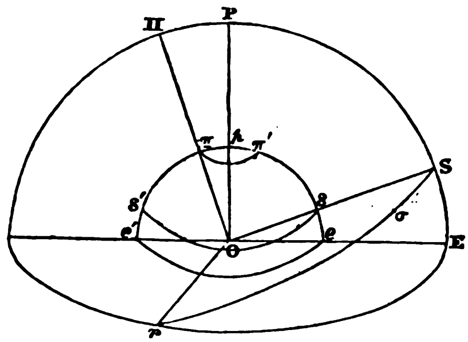
Suppose the sun to be at ♈, which will happen at the time of year called the *vernal equinox*, and suppose it to be moving in the ecliptic towards $S$: for three months its distance from the equator will increase, and at the end of that time it will be at $S$, a point in the great circle passing through $P$ and $\Pi$. Its distance from the equator will then diminish, until at the end of three months it will again be in the equator; this will happen at the *autumnal equinox.* The sun will now go to the south of the equator, and at the end of the next three months will be at $S'$ on the great circle passing through $P$ and $\Pi$. Lastly, after passing $S'$ the sun will again approach the equator, and at the end of three months more will be again at ♈.

16. For a few days before and after passing the points $S$ and $S'$, the sun will move nearly parallel to the equator, and therefore, will neither approach it nor recede from it; hence, so far as motion to or from the equator is concerned, the sun may be said at those points to be stationary for a short period, and they are on this account called the *solstices.* $S$ is the *summer* solstice, $S'$ the *winter* solstice; the great circle passing through the solstices and the poles of the equator and ecliptic, is called the *solstitial colure.*

## ON CLIMATE.

17. It will be easily seen, that all parts of the earth's surface are not equally affected by the sun's heat; the term *climate* is used to express the difference amongst the several regions of the earth in this respect.

Let $O$ be the centre of the earth; ♈ $S$ the ecliptic, ♈ $E$ the equator, $\Pi$, $P$, their respective poles; and let $\pi$, $p$, $s$, $e$,




be the points in which the lines $O\Pi$, $OP$, $OS$, $OE$, respectively meet the earth's surface; also $\pi\pi'$, $ss'$, small circles on the earth's surface made by planes parallel to the equator, and $ee'$ the great circle in which the plane of the equator cuts the earth.

Then the portion of the earth's surface to the north of $\pi\pi'$, and an equal portion to the south of a similar small circle round the south pole, are called the *frigid Zones:* the portion between $\pi\pi'$ and $ss'$, and a similar portion in the southern hemisphere, are called the *temperate Zones:* and the portion between $ss'$ and a similar circle in the southern hemisphere are called the *torrid Zones.*

The small circle $\pi\pi'$ is called the *Arctic Circle*, and a similar one in the southern hemisphere the *Antarctic Circle.* The small circle $ss'$ is called the *Tropic of Cancer*, and a similar one in the southern hemisphere the *Tropic of Capricorn;* because the sun, after receding from the equator,

turns ($\tau\rho\acute{\epsilon}\pi\epsilon\iota$) on entering the signs of Cancer and Capricorn, and again approaches the equator.

18. The peculiarity of the torrid zone is that a place situated within it will have the sun *vertical*, that is, exactly in its zenith, twice in the year. For let $\sigma$ be the place of the sun at any given time, then, if we join $O\sigma$, this line will manifestly intersect the earth's surface at some point between the tropics, and that place, with all others in the same latitude, will have the sun in the zenith; and the same thing will take place when the sun has passed the solstice by a distance equal to $\sigma S$. Even when the sun is not exactly vertical, its rays fall with a smaller obliquity on places between the tropics than in the temperate zones; hence the extreme heat of tropical climates.

The temperature of the different countries of the earth does not however depend wholly upon their latitude, but upon a variety of local circumstances. This will be at once evident from inspection of a map upon which are laid down a system of *isothermal lines**, or lines passing through all places, the mean temperature of which throughout the year is found by observation to be the same. It will be seen for instance from inspection of such a map that the isothermal line which passes through the centre of Great Britain passes the neighbourhood of New York in a latitude of little more than 40°, and that the same line passes through the north of the Black Sea in latitude 45°; thus the mean temperature enjoyed by England is the same as that of some places 10° or 15° nearer to the equator. The two principal causes of these local differences of climate are the influence of prevalent warm winds, and that of warm ocean currents; to the latter cause the peculiar warmth of England with reference to its latitude is mainly due. The climate of a country will also vary sensibly with the condition of the surface of the earth, and will be favourably influenced by the clearing of forests, the draining of swamps, and the like.

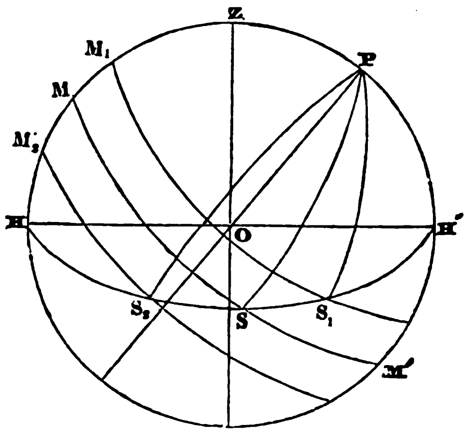
The peculiarity of the frigid zones will be noticed in Article 20.

* See such a map in Johnstone's Physical Atlas.

## ON THE LENGTH OF THE DAY.

19. The motion of the sun in the ecliptic during one day is not very great; hence in considering the effect of its motion on the length of the day it will be sufficient for the purpose of explanation, to suppose it to preserve the same position in the ecliptic during one revolution of the earth, or during one of its own apparent revolutions about the earth.

Let $O$ be the earth's centre, $Z$ the zenith of a place on its surface, $HH'$ the horizon of the place, $MSM'$ the equator,




$P$ its pole. Then we can determine the length of the day for any given position of the sun in the ecliptic, by supposing it to describe a small circle in a plane perpendicular to $OP$; as long as it is above the horizon it is day, the remainder of the twenty-four hours is night. We shall consider three cases.

(1) Let the sun be in the equator; then its diurnal path will be the great circle $SM$, and if we join $PS$ by an arc of a great circle, half the day will be measured by the angle $MPS$ (called an *hour-angle*). But it is not difficult to see that the angle $MPS$ is a right angle, hence the hour-angle measuring the length of the day is two right angles; consequently that which measures the length of the night must be two right

angles, or the day and night are equal. When therefore the sun is in the equator, day and night are equal all over the world; hence the equator is sometimes called the *equinoctial line*, and the first points of Aries and Libra are called the *equinoxes*.

(2) Suppose the sun to be in the summer solstice; then its diurnal path will be the small circle $S_1M_1$, the arc $MM_1$ being that which measures the obliquity of the ecliptic. Join $S_1P$ by an arc of a great circle; then the hour-angle $S_1PM_1$ measures half the day, and this angle is greater than a right angle, hence the days are longer than the nights to places in northern latitudes.

(3) In like manner, if the sun is in the winter solstice, and $S_2M_2$ its diurnal path, the hour-angle $S_2PM_2$ will measure half the day, and the days will be shorter than the nights.

For intermediate positions of the sun the results will be easily inferred.

20. Let us consider the peculiarities of day and night in the frigid zones. Let us suppose $PZ$ to be equal to the obliquity of the ecliptic, then the small circle $S_1M_1$ will pass through $H'$, that is to say, when the sun is in the summer solstice it just does not set to a place upon the Arctic Circle; and in general, in order that the sun may set to any given place, its angular distance from the Pole (or *North Polar distance*, as it is called,) must be greater than the latitude of the place; suppose, for instance, we take a place in latitude 70°, then the sun will not set to that place from the time that its north polar distance is 70°, until after having passed the solstice its north polar distance is 70° again. Corresponding to those long summer-days there will be equally long winter-nights; and at the poles there will be a day of six months in length, succeeded by a night equally long.

The results of this and the preceding article will probably receive elucidation from reference to the figure of Art. 14; in that article the earth has been represented as occupying different positions, the sun remaining fixed, here we have

supposed the earth stationary, and have supposed the sun to move in the ecliptic.

Consider the position $A$ of the earth in Art. 14, then it is evident that if the earth be supposed to revolve about its axis $POP'$, any place upon its surface will in the course of a revolution be during equal times in light and in shade, and the same holds true of the position $C$; these two positions correspond to the sun being in the vernal and autumnal equinoxes respectively. Again, consider the position $B$, and suppose the earth to revolve on its axis as before; then it is evident from inspection that any point in north latitude, such as $L$, will perform a longer path in light than in shadow, and the contrary will be the case for the position of $D$ of the earth; in other words, places in north latitude will have the day longer than the night when the sun is in the summer solstice, and the reverse for the winter solstice. It will be seen moreover that any point between $P$ and $B$ is in perpetual day, and any point between $P$ and $D$ in perpetual night.

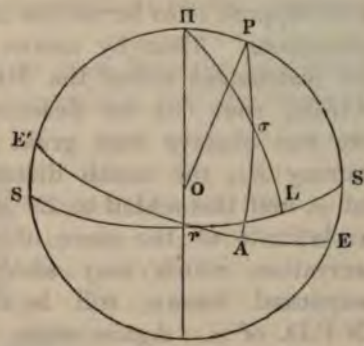
21. It may be noticed that the most refined calculations prove, that throughout the centuries to which the records of astronomical observations extend the length of the mean day has been invariable; in other words, the velocity of rotation of the earth upon its axis has remained sensibly the same.

## ON THE MODE OF DETERMINING THE PLACE OF A HEAVENLY BODY.

22. This will be done on the general principle explained in Art. 6, and already applied to the case of terrestrial latitude and longitude in Art. 10.

Let $O$ be the centre of the celestial sphere, $E \Upsilon E'$ the equator, $S \Upsilon S'$ the ecliptic, $P$, $\Pi$ their respective poles. Let $\sigma$ be any heavenly body, the position of which we desire to determine: draw through $\sigma$ the arcs of great circles $P\sigma A$, $\Pi\sigma L$; then $\Upsilon A$ is called the *Right Ascension*, $A\sigma$ the *Decli-*

*nation* of $\sigma$, and if these be given the position of $\sigma$ will be determined. It will be the same thing if we suppose $P\sigma$, the North Polar distance, to be given instead of the declination. The Right Ascension is measured from ♈ in the direction of the sun's motion. Right Ascension is usually written in the abbreviated form R.A., and North Polar Distance, N.P.D.




23. The position of $\sigma$ may be equally well determined by means of the arcs ♈$L$ and $L\sigma$, which are called respectively its *longitude* and *latitude*. Care must be taken not to confuse these terms with the same as applied to places upon the earth's surface.

24. By measuring R.A. and longitude from ♈, we appear to assume that ♈ is a fixed point. This is not accurately true, as we shall see hereafter; its motion however is sufficiently slow to allow us, in general, to conceive it to be fixed; in making observations the motion is allowed for.

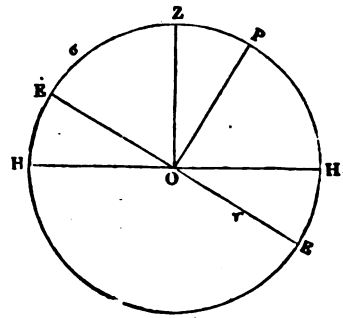
## ON THE MODE OF MAKING OBSERVATIONS OF THE HEAVENLY BODIES.

25. We propose to explain the mode in which the Right Ascension and Declination of a star become matters of observation, and to describe the principal instruments by means of which the observations are made.

Let $Z$ be the zenith of the place of observation, $O$ the centre of the celestial sphere, $E$♈$E'$ the equator, $P$ its pole, $HZP$ the meridian of the place.

Let $\sigma$ be the star or other heavenly body, the R.A. and declination, (or N.P.D.) of which we wish to determine,

and suppose it to be on the meridian when we make our observations. Then by means of an instrument called the Mural Circle, soon to be described, we can observe with great accuracy $Z\sigma$, the zenith distance of $\sigma$, and this added to $ZP$, the co-latitude of the place of observation, which may also be supposed known, will be the N.P.D. of $\sigma$. Again, since the whole heavens turn about $OP$

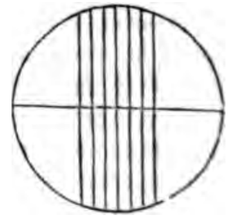



uniformly in 24 hours, called *sidereal* hours, if we note the time of ♈ passing the meridian and that of $\sigma$, the difference between these times will measure the arc ♈$E$, or the R.A. of $\sigma$; for instance, suppose the difference of time to be 1 hour, then the R.A. would be $\frac{360^\circ}{24}$, or $15^\circ$. The method adopted in practice is to have a clock which indicates $0^h\,0^m$ when ♈ is on the meridian, and then the sidereal time of a star's passing the meridian, or the time of its *transit*, converted into degrees at the rate of $15^\circ$ to 1 hour, will be the R.A. of the star.

Thus the determination of the R.A. and N.P.D. of a heavenly body is reduced to that of the time of passing the meridian, and the meridian zenith distance. We shall proceed to describe the Transit Instrument and the Mural Circle, by means of which this is effected.

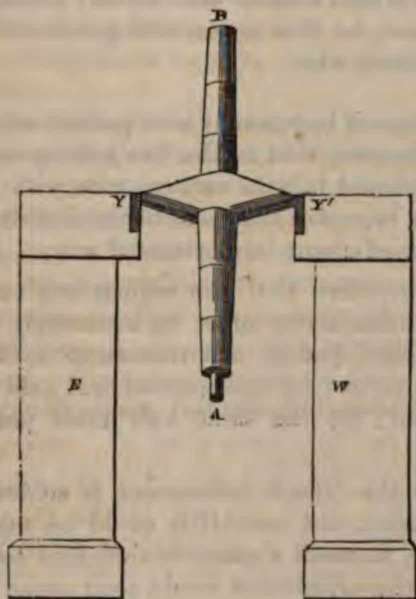
26. *The Transit Instrument.*

This instrument consists of a common astronomical telescope, fixed firmly to an arm $YY'$, the extremities of which are cylindrical and are supported by two pillars of solid masonry $E$, $W$, so placed that the arm $YY'$ points east and west. In the focus of the object-glass are placed a certain number of fixed vertical wires, usually five, or seven, and one horizontal wire



passing through the centre of the field, as in the annexed figure.

The telescope moves, as will be seen from the preceding description, in the plane of the meridian, and the mode of




observation is as follows: Before the object to be observed comes upon the meridian, the transit instrument is set in such a position that the object shall pass as near as possible through the centre of the field of view: this may be done by means of a variety of contrivances, but it requires an approximate value of the zenith distance of the object, which however may be supposed known from previous observations sufficiently nearly for the purpose, in the case of all bodies of which it is necessary to observe the transit. The time of transit is the moment at which the object crosses the middle vertical wire, supposing that wire to be in perfect adjustment; but to avoid the error of imperfect adjustment, and also to diminish as far as possible the errors of observation, it is usual to note the time of transit across each of the vertical wires, and take the mean of these observed times as the true time of transit.

The observation requires the assistance of a sidereal clock which beats seconds in a very distinct manner; the observer before looking into the telescope takes notice of the time indicated by the clock, and by counting the beats of the pendulum knows the time which elapses afterwards: having a book and pencil in his hand, he thus notes with great accuracy the time of transit over each wire.

27. The transit instrument is in perfect adjustment when the *line of collimation*, that is, the line joining the intersection of the horizontal and middle vertical wire with the centre of the object-glass, moves in the plane of the meridian, the instrument being turned about its horizontal axis.

In order therefore that the adjustment may be perfect, (1) the line of collimation must be accurately perpendicular to the geometrical line in the transverse axis about which the instrument turns; (2) this geometrical axis must be accurately horizontal; (3) the same axis must point accurately east and west.

In practice the transit instrument is seldom or never in perfect adjustment, and even if it could be made perfect at any given time, so small a cause is sufficient to sensibly disarrange it that the adjustment would soon cease to be perfect. The three errors therefore corresponding to the three adjustments above mentioned, and which are known as the errors of *collimation, level,* and *deviation,* respectively, are ascertained, and correction made in the observed time of transit. For the method of determining these errors and correcting the time of transit, reference must be made to a more complete treatise on Astronomy.

28. *The Sidereal Clock.*

The clock when properly adjusted ought to indicate $0^h\,0^m\,0^s$ when the first point of Aries is on the meridian, and should indicate the lapse of 24 *sidereal* hours during the interval of two successive transits of that point. The time of the transit of a star according to the sidereal clock will be the star's R.A. in *time*.

The clock is corrected by observing the time of transit of a star, the R.A. of which is accurately known, and comparing the known R.A. with that indicated by the clock.

By observing two transits of the same star, or the transit of two known stars, we can ascertain the clock's *rate*, that is, the rate at which it is gaining or losing.

And by observing three transits, we can ascertain whether its rate is regular.

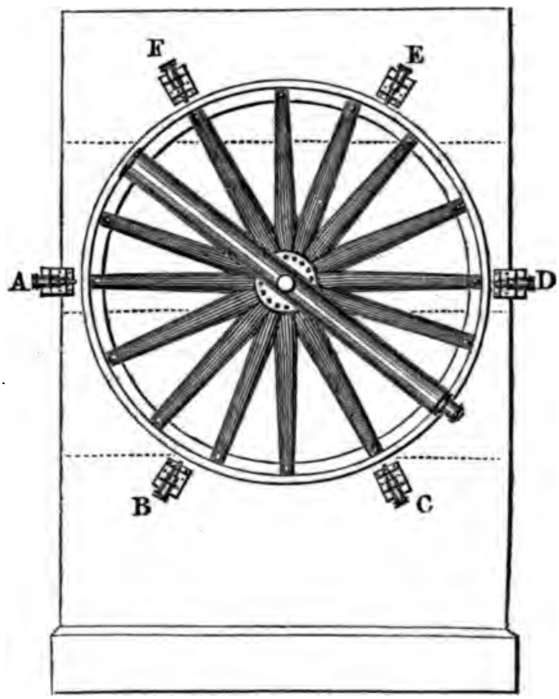
It would take us beyond the scope of the present treatise to describe accurately the method of setting the clock, that is, determining the precise position of the first point of Aries. Of course if we could make an observation of the sun at the moment when its declination was zero, the end would be accomplished; this is evidently impracticable; we may however make a series of observations when the declination is very small, and when it changes from northern to southern, and by means of certain precautions we may thus ascertain with considerable accuracy the time at which the sun actually crossed the equator. In the present state of astronomy the error and rate of the clock are both ascertained by help of known stars; the places of 100 stars which have been observed unremittingly during the greater part of a century are given for the purpose in the Nautical Almanack.

29. *The Mural Circle.*

This instrument consists of an astronomical telescope firmly clamped to a flat circular rim, which moves about an axis passing into a strong vertical wall to which the back of the instrument is applied. The graduation is made on the side of the rim perpendicular to the wall, and the reading off is effected by means of microscopes $A$, $B$, $C$, $D$, $E$, $F$. In the focus of the object-glass are two fixed wires, one vertical and one horizontal, and besides these there is a moveable horizontal *micrometer* wire, the nature of which will require some explanation.

A micrometer wire is one which is moveable by means of a screw within the observer's reach, the head of which is graduated and so contrived as to indicate the distance through which the wire is moved. Suppose, for instance, in the above

case, when the micrometer wire coincides with the fixed horizontal wire, the pointer on the screw-head indicates $0^0\ 0'\ 0''$, then if when the wire is made to coincide with any given




object in the field of view the reading of the screw-head is $n''$, we shall know that the distance between the image and the fixed horizontal wire subtends an angle of $n''$ at the centre of the object-glass, in other words, that the angular distance between the object and the centre of the field of view as measured by an arc of a great circle on the celestial sphere is $n''$.

30. Since the purpose of this instrument is to observe the zenith distance of heavenly bodies as they pass the meridian, the line of collimation ought to move in the plane of the

meridian. But it is not difficult to see that the error occasioned by a slight deviation of the line of collimation from that plane will not be so important as in the case of a transit instrument: the error of principal importance is this; when the instrument is pointed to the zenith, one of the microscopes ought to indicate accurately $0''$, and this will not generally be the case, the error in the reading may be called the error of collimation in altitude, and may either be calculated and allowed for, or may be got rid of entirely by a method of observation which we shall presently describe. Properly speaking, the error which we have called that of collimation in altitude consists of two, one arising from the fact that if the line of collimation were in perfect adjustment there would in general be an *index error*, and the other from the usual want of accurate adjustment of the line of collimation; these two however are necessarily joined together, and may be conveniently spoken of as one error.

31. The reading off is made with six microscopes instead of one, for the sake of greater accuracy, especially in these respects: we diminish the probability of error from defective graduation; but still more we avoid any error arising from false centering, i.e. from the centre about which the circle turns not coinciding with the actual centre of the graduated rim; for it will be easily seen that if on this account any microscope give a reading in excess, the opposite one will give a reading as much in defect, and thus the mean of the two will be correct. The mode of reading by means of a microscope is simply this: the part of the graduated rim, the image of which coincides with the centre of the field of the microscope, is that the position of which we desire to determine; but this will in general not happen to coincide with a line of graduation; we have therefore to determine the distance of the centre of the field from the nearest line of graduation. This is done by means of a micrometer wire, which when it coincides with the centre of the field gives a reading on the screw-head of $0''$, and hence if we turn the screw-head until the micrometer wire coincides with the image of the nearest line of graduation, the reading on the screw-head

will be the quantity to be added to the reading of the circle, corresponding to the nearest line in question. The first reading is thus made with the naked eye, and the microscope then furnishes a correction to this reading.

32. The mode of observation with the mural circle is as follows.

Before the star, or other heavenly body to be observed, crosses the meridian, let the telescope be directed downwards towards a trough of mercury, in such a manner that the image of the star seen by reflexion at the surface of the mercury may pass through the field of view: this can be done, because the zenith distance is supposed to be known approximately. Let the circle be clamped in this position, and the microscopes read.

When the star, as seen by reflexion, comes into the field of view, let it be bisected by the moveable micrometer wire, a short time before it reaches the middle of the field.

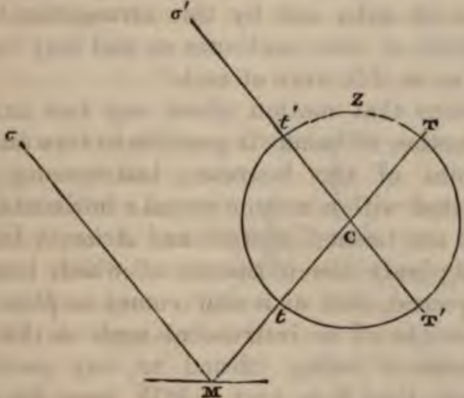
Let the circle be now unclamped, and turned rapidly round until the star is visible by direct vision, and by means of a screw provided for the purpose let the circle be slightly moved until the star is bisected by the fixed horizontal wire. This may be effected by a skilful observer soon after the star has passed the middle of the field.

The microscopes may now be read, and the mean of the six readings will be the result of the observation by direct vision. The micrometer may also be read, and this reading added to the mean of the readings of the six microscopes, as examined before the observation, will give the result of the observation by reflexion.

The mean of the two observations will give the true altitude or zenith distance of the star, free from error of collimation in altitude. That this will be so may be seen as follows.

33. Let $TtM$ be the position of the line of collimation, when the star $\sigma$ is seen by reflexion. Then if $Z$ be the zenith point, the reading ought to be $Zt$, but in consequence of the collimation error let it be $Zt + C$. Also let $T't'$ be

the position of the line of collimation, when the star is seen by direct vision, and the reading will be $Zt' + C$; hence




the mean of the readings $= \dfrac{Zt - Zt'}{2} = \dfrac{tt'}{2} =$ the true altitude of the star.

34. The Transit Instrument and the Mural Circle are the instruments of daily use in observatories, and we recommend the student, if it should be within his power, to visit an observatory for the purpose of inspection of the instruments.

Other instruments are used, especially the Equatorial, which for many purposes of Astronomy is indispensable, inasmuch as the Transit Instrument and Mural Circle are applicable only in the case of bodies which can be observed upon the Meridian. In the Equatorial the telescope is attached to a framework which revolves about an axis parallel to the axis of the earth, so that if the telescope be directed to a star and the instrument made to revolve about this *polar axis* the star may be made by this single motion to remain in the field of view: but again, supposing the instrument to be at rest so far as revolution about the polar axis is concerned, the telescope has a motion in *declination*, that is, it can revolve about an axis perpendicular to the polar axis. By the combination of these two motions it is clear that the telescope may be directed to any part of the heavens. The

best instruments of this construction are fitted with a clock-work apparatus, which causes the equatorial to revolve about the polar axis at the same rate as that with which the earth revolves upon its axis, and by this arrangement an object once in the field of view continues so and may be examined by the observer as if it were at rest.

It is obvious that motion about any two axes at right angles to each other will make it possible to turn an instrument upon any point of the heavens; instruments are sometimes constructed with a motion round a horizontal and a vertical axis, and are termed *Altitude* and *Azimuth* Instruments.

A heavenly body the existence of which has only been recently discovered, such as a new comet or planet, must be observed by means of an instrument such as the equatorial, which is capable of being turned to any portion of the heavens; when the R.A. and N.P.D. have by this means been determined with some degree of accuracy, the place of the body may be ascertained more nearly by means of meridian observations made with the Transit Instrument and Mural Circle, for the application of which it will be remembered that an approximate knowledge of the place of the body to be observed is indispensable.

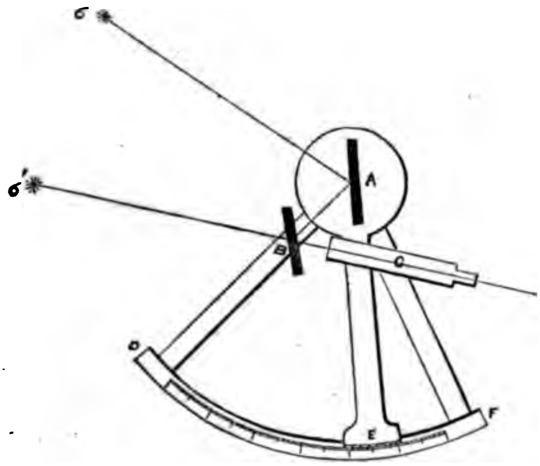
We shall conclude this part of the subject by describing Hadley's Sextant, an instrument of great importance on account of its applicability to nautical purposes; the instruments already described, and all others which require fixed supports, are useless at sea on account of the constant motion of the vessel.

### 35. *Hadley's Sextant**.

The principle on which this instrument is constructed, depends upon a proposition proved in Optics, (Art. 27, p. 481), viz. that when a ray of light is reflected at two mirrors, the angle of deviation is equal to twice the angle between the mirrors.

* Commonly called Hadley's, though the priority of invention undoubtedly belongs to Newton. Newton communicated it in a letter to Dr Halley, amongst whose papers the description of the instrument was found after his death in Newton's own handwriting. Hadley's invention however was very possibly independent of Newton's.

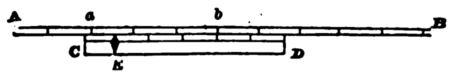
The figure represents Hadley's Sextant; $DF$ is a portion of a graduated rim, the graduation, as the name imports,




extending usually to about one sixth of a circumference, but sometimes to more. $AE$ is a moveable radius of the circle, which carries with it a small mirror $A$ of silvered glass at one end, and a vernier at the other*. $B$ is a piece of glass silvered

* The *Vernier* is a contrivance for reading off more accurately than is possible with a simple pointer.

Let $AB$ be a portion of the graduated limb of any instrument, and suppose that instead of a simple pointer we have a small piece of brass $CD$, having apointer $E$ and




the space between $E$ and $D$ graduated in such a manner that $n$ of its graduations shall be equal to $n-1$ of those on the limb $AB$.

Let $a$, $\beta$ be the lengths of the respective graduations;

$$\therefore\ na=(n-1)\beta, \qquad \text{or } \beta-a=\frac{\beta}{n}.$$

Now suppose that after an observation $E$ does not point accurately to any line of graduation of the instrument; let $a$ be the nearest, then we must add to the reading at $a$ the distance between $a$ and the pointer. Suppose the $r^{\text{th}}$ division of the vernier almost exactly points to a line of graduation, as at $b$; then the quantity to be added to the reading at $a$

$$=r\beta-ra=r\,\frac{\beta}{n}.$$

Suppose, for instance, that $\beta=1^\circ$, and $n=6$, then $\frac{\beta}{n}=10'$; and if $r=4$, the quantity to be added to the reading at $a=40'$.

over half its surface, and is so fixed that when the reading of the vernier is zero, the surfaces of the mirrors *A* and *B* are parallel. *C* is a small telescope attached to the instrument, and so arranged that its axis passes through the line of division between the silvered and unsilvered parts of the glass *B*.

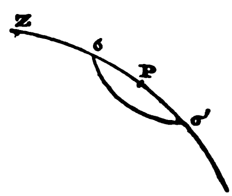
To find the angle subtended by the line joining two objects $\sigma$, $\sigma'$, let the instrument be held in such a position, and the moveable radius so adapted, that the image of $\sigma$ as seen by reflexion at the two mirrors, coincides with that of $\sigma'$, as seen by direct vision; then the angle between the two objects will be twice the angle between the mirrors, or twice the arc between the pointer of the vernier and the zero point of the instrument, since when the mirrors were parallel the reading of the vernier was zero. Consequently, if we graduate the arc *DF* in such a manner as to make one of its divisions correspond to $2^0$, the reading given by the vernier will be the angle required.

By means of this instrument the angular distance between two objects, or the altitude of a heavenly body above the horizon, may be ascertained with sufficient exactness for nautical purposes.

## ON METHODS OF FINDING THE LATITUDE OF A PLACE.

36. In our explanation of the mode of finding the N.P.D. of a heavenly body, we have assumed the latitude of the place of observation to be known; we proceed to shew how it may be ascertained.

Let $\sigma\sigma'$ be the small circle described by a circumpolar star, round the pole $P$, in consequence of the diurnal revolution of the heavens; $Z$ the zenith of the place of observation, $Z\sigma P\sigma'$ its meridian.




When the star is on the meridian at its upper transit, let its zenith distance $Z\sigma$ be observed; and when it is again on

the meridian at its lower transit, let its zenith distance $Z\sigma$ be observed.

$$\text{Then} \qquad ZP = \frac{Z\sigma' + Z\sigma}{2} = \text{the co-latitude;}$$

hence the latitude is known.

The preceding method is applicable only when we have a fixed instrument in the plane of the meridian, as in an Observatory. Methods applicable to observations made at sea require mathematical calculations, into which it is not the purpose of this treatise to enter.

## ON THE FIXED STARS.

37. If we look at the heavens night after night, we shall easily conclude that the greater part of the stars preserve, at least approximately, the same relative positions, and if we determine their R.A. and N.P.D., we find them nearly constant; such stars are called *fixed* stars, a name which distinguishes them from the *planetary* bodies, of which we shall afterwards have to speak more particularly. The modes of determining the R.A. and N.P.D. of a star, which we have described, will require considerable correction, but, supposing that we have the means of determining accurately these elements, we can make a catalogue of the fixed stars. Such catalogues have been made, and are of the utmost use in practical Astronomy; the place of a star becomes known accurately only by a long series of observations, and when the R.A. and N.P.D. of a star have thus been satisfactorily established, it is called a *known* star. These fixed stars are at an enormous distance from the earth, as is at once concluded from this fact, that the most powerful telescope does not exhibit any sensible disk; but of the greatness of their distance we shall presently have a more definite notion, when we come to speak of parallax. The stars are divided into groups called constellations; this however is not done upon any good and convenient system, but in a manner apparently quite arbitrary, and certainly very fanciful. If we refer the

places of the stars, and those of the sun, moon, and planets, to the surface of the celestial sphere, the stars will be fixed in position, and the sun, moon and planets will move amongst them; hence we may speak of the motion of the sun amongst the fixed stars; and the signs of the zodiac are in fact the names of twelve constellations, which at the time the names were given were coincident in position with the signs bearing the same names, but now, for reasons to be hereafter assigned, occupy other positions.

The latitude and longitude of a star may be deduced from its right ascension and declination, and stars may be catalogued accordingly; but this method is not so convenient as that of registering their right ascensions and declinations.

## ON THE CORRECTIONS OF ASTRONOMICAL OBSERVATIONS.

38. We have described the mode of determining by observation the R.A. and N.P.D. of any heavenly body, but the observations so made require several important corrections, which we now proceed to explain.

## ON REFRACTION.

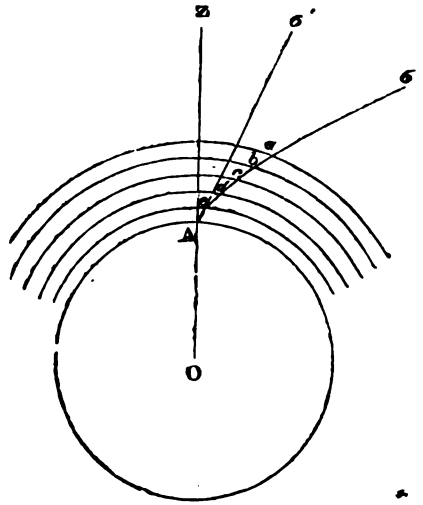
39. The first correction is due to the deviation caused in a ray of light by its passage through the atmosphere.

Let $O$ be the centre of the earth, $A$ the place of observation, $Z$ its zenith. And conceive the atmosphere to be formed of concentric strata of air, diminishing in density as they are further from the earth's surface.

Then a ray of light in passing through the atmosphere will, on entering each stratum, be bent towards the normal, (Art. 7, p. 465), and consequently the ray of light by which the object $\sigma$ is seen by an observer at $A$ will be of the kind represented by the line $\sigma abcdeA$. In the limit, when we consider the strata to be indefinitely thin, the path of the ray through the atmosphere will be a continuous curve, and the

direction in which the object will be seen will be the tangent to the curve at the point nearest the observer's eye.

The atmosphere will be exactly similar on opposite sides of the vertical plane through $ZA\sigma$, and therefore the ray of




light will not be refracted out of that plane. Hence a heavenly body appears to be raised in a vertical plane above its true position, and the zenith distance given by observation will be too small; the amount of correction to be applied depends upon calculations into which we shall not enter.

It may be mentioned however that of all the corrections, which it is necessary to apply to the observed places of heavenly bodies, this is the most difficult to determine with accuracy. The observations of the transit instrument are obviously not affected by it; but in the case of the mural circle it is a very serious error, especially when the zenith distance is considerable; for, without inquiring into the actual law of its variation, it is evident that the effect of refraction is zero when the heavenly body is in the zenith, and increases as the zenith distance increases. If the atmosphere were homogeneous there would be no difficulty in assigning the law of refraction, but this is not the case, and the actual consti-

tution of the atmosphere being unknown there is a consequent difficulty in determining the refraction; mathematicians have however been able to construct tables, which represent sufficiently nearly the amount of the error provided the body observed be not very near the horizon. These tables are constructed for a certain standard height of the barometer, and a certain standard temperature of the air; the corrections corresponding to other states of the atmosphere may be inferred, and for this purpose it is necessary to note the height of the barometer and of the thermometer in making an observation with the mural circle.

The effect of refraction upon the appearance of the sun and moon when very near the horizon is worthy of notice. We have seen that the effect is in general to raise a heavenly body above its true position, but this effect will take place in different degrees for different points upon the surface of a body of considerable disk, such as the sun or moon, especially when those bodies are very near the horizon, in which case the effect of refraction is greatest. If we consider then the vertical diameter of the disk of the setting sun, the highest and lowest points will both be raised by refraction, but the lowest more than the highest, consequently the diameter will be shortened; and the same thing will be true of every line parallel to this diameter; consequently the circular disk of the sun will appear as an oval, the oval however being more flattened at the lower than at the upper part of the disk. This is a result capable of being verified by common observation. It may be noticed that the apparent increase of the magnitude of the disk of the sun or moon when near the horizon is not connected with refraction, and is in fact a mere illusion, arising from the fact that when these bodies appear in the horizon, we judge of them as terrestrial objects, whereas when seen above in the heavens the judgment is bewildered and forms no definite conclusion as to their magnitude. The angular magnitude of the moon is in fact materially less when in the horizon than when seen at a great altitude.

40. *On the phænomenon of Twilight.*

The consideration of the effect of refraction upon the

directions of rays of light passing through the atmosphere, renders this a convenient place to explain the phænomenon of twilight. We have shewn in the preceding article, that the heavenly bodies are apparently raised in a vertical plane by the refraction of the atmosphere, and it follows that they become visible on the earth's surface when they are in fact somewhat below the observer's horizon. But before it becomes visible, the rays of light from the sun below the horizon illuminate the atmosphere, which to some extent reflects and scatters the rays in all directions, and the result is a faint light which precedes the rising of the sun and follows its setting, and which we call *twilight*. Twilight begins and terminates when the sun is about 18° below the horizon; but its duration manifestly varies with the latitude, for the time which is required for the sun to rise through 18° vertically depends upon the inclination of its diurnal path to the horizon of the place, and is less as this inclination is greater, that is, as the place is nearer to the equator. To take an extreme case, suppose the place to be on the equator, and for simplicity's sake, suppose the sun to be in ♈, then its diurnal course will be a vertical great circle, and the duration of morning or evening twilight will be the time taken to describe 18°, or not much more than an hour.

In more northern latitudes the twilight may endure all night: it is not difficult to determine the conditions under which this will take place.


Let $Z$ be the zenith of the place, $H'H$ the horizon, $P$ the pole of the equator, $S$ the sun at midnight. Then the condition of twilight lasting all night is, that the greatest depression of the sun below the horizon shall not be more than 18°, or $HS$ not greater than 18°.

Let $l$ be the latitude of the place $= HP$,

$\delta$ the sun's declination $= 90° - PS$:

then $$HS = PS - HP = 90° - \delta - l;$$

therefore we must have

$$90^\circ - \delta - l \text{ not} > 18^\circ,$$

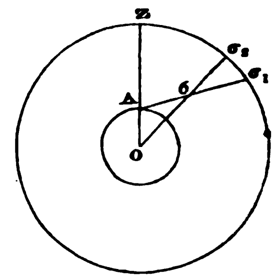
$$\text{or} \quad \delta + l \text{ not} < 72^\circ.$$

For instance, the latitude of London is $51\frac{1}{2}^\circ$, therefore twilight endures all night so long as the sun's declination is not less than $20\frac{1}{2}^\circ$, that is, from the latter end of May to the latter end of July.

## ON PARALLAX.

41. This second correction is rendered necessary by the fact of our observations being made at the surface of the earth, and not at the centre. If the position of a heavenly body were registered according to its apparent position as seen from a certain place, the position so determined would not agree with that determined by another observer at a different place on the earth's surface; a correction is therefore applied to observations, such as shall reduce them to what they would have been if made from the earth's centre. This we proceed to explain more particularly.

Let $O$ be the earth's centre, $A$ the position of an observer on its surface, $Z$ the zenith; and let $\sigma$ be a heavenly body, which an observer at $A$ will refer to the point $\sigma_1$ on a sphere described with centre $O$, and an observer at $O$ will refer to $\sigma_2$. Hence the zenith distance of $\sigma$ as observed from $A$ will have to be diminished by the quantity $\sigma_1 \sigma_2$ in order to reduce it to what it would have been if observed from $O$. Parallax, it is evident, takes place in a vertical plane, and depresses a heavenly body, which is exactly the opposite effect to that of refraction; in speaking, however, of parallax as compared with refraction, it is to be carefully borne in mind that they are corrections in very different meanings of the word, for, in consequence of refraction, the object is not actually in the



position in which it seems to be, whereas the correction for parallax is merely a reduction of the observations made at one place to what they would have been if made at another.

42. In the figure let $p = A\sigma O = \sigma_1\sigma_2$ nearly, $ZA\sigma = z$, then, from the triangle $A\sigma O$,

$$\sin p = \frac{AO}{O\sigma} \sin z;$$

or since $p$ is very small, if we use the circular measure,

$$p = \frac{AO}{O\sigma} \sin z.$$

This quantity $p$, since it varies with the altitude of the body, is called the *diurnal* parallax; its greatest value is when the body is in the horizon or $z = 90^\circ$, in which case, (if we call the value $P$),

$$P = \frac{AO}{O\sigma}, \text{ and } p = P \sin z;$$

$P$ is called the *horizontal* parallax.

43. It is found that some of the heavenly bodies, namely, the fixed stars, have no sensible horizontal parallax, but for the sun, moon, and the planets (of which we shall hereafter speak more particularly), the value is sensible, and the determination of its value in each case becomes a matter of great moment; for it will be seen, that the knowledge of the horizontal parallax of a heavenly body informs us at once of the distance of that body from the earth's centre in terms of the earth's radius, the magnitude of which we shall presently shew how to find. In fact, we have by the last article

$$O\sigma = \frac{AO}{P};$$

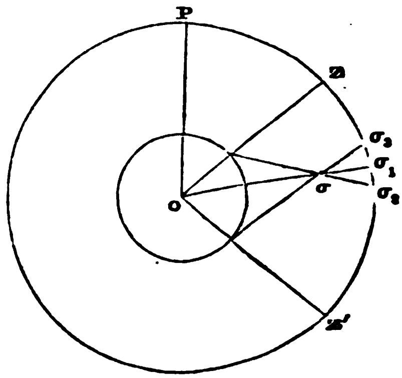
and if $R$ be the earth's radius, $x$ the distance of the heavenly body, and the horizontal parallax be given in seconds,

$$x = \frac{180 \times 60 \times 60}{P \times 3.14159} R. \quad \text{(Art. 55, p. 153.)}$$

The value of the sun's horizontal parallax is about 8.5776", that of the moon's 57'4".

44. *To find the horizontal parallax of a heavenly body by observation.*

Let $ZZ'$ be the zeniths of two places on the same meridian $PZZ'$: and let $\sigma$ be a heavenly body upon the meridian, which is referred by observers at the two places to the points $\sigma_2$ $\sigma_3$ on the celestial sphere respectively, its true position as seen from the centre of the earth being $\sigma_1$. Then if $P$ be the horizontal parallax of $\sigma$, we have




$$\sigma_1 \sigma_2 = P \sin Z\sigma_2,$$

$$\sigma_1 \sigma_3 = P \sin Z'\sigma_3,$$

also $$Z\sigma_2 - \sigma_1\sigma_2 + Z'\sigma_3 - \sigma_1\sigma_3 = ZZ',$$

$$\therefore \quad P = \frac{Z\sigma_2 + Z'\sigma_3 - ZZ'}{\sin Z\sigma_2 + \sin Z'\sigma_3}.$$

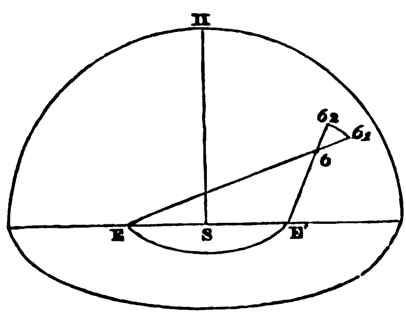
In this equation $Z\sigma_2$, $Z'\sigma_3$ are known by observation, and $ZZ'$ is the difference of the latitudes of the places, (or the *sum*, if the places are on opposite sides of the equator, as in the figure;) hence $P$ is completely determined.

We have spoken of the two places of observation being upon the same meridian; practically this may be an inconvenient or an impossible condition; it is desirable, however, that the condition should be satisfied as nearly as possible, since the motion of the moon is so rapid, that if there be any considerable difference in longitude between the two places, the change of the moon's declination in its passage from one meridian to the other will be very sensible; and to apply a correction for this change of declination would introduce the

uncertainty which attaches to the supposition of an extremely accurate knowledge of the moon's motion. The observatories of Greenwich and the Cape of Good Hope answer the conditions of the problem exceedingly well, their distance in latitude being more than 85° and their difference in longitude not much more than 18°.

This method is applicable to the moon, the parallax of which is considerable; but the sun's parallax, being a very minute quantity, can only be satisfactorily determined by the help of very particular phænomena, a description of which we shall not here enter upon*. It may be stated however that the value above given probably does not differ from the truth by more than one two-hundredth part.

45. The diurnal parallax of the fixed stars is found to be wholly insensible: but there is another kind of parallax due to the change of position of the earth in its orbit, which is called *annual* parallax, and it becomes a question how far this will affect the apparent position of a fixed star.




Let $S$ be the sun, $E$, $E'$ two positions of the earth on opposite sides of it; $\sigma$ a star which an observer on the earth at $E$ refers to the point $\sigma_1$, and an observer on the earth at $E'$ to the point $\sigma_2$ on the celestial sphere; then $E\sigma E'$ is the greatest variation in the position of $\sigma$ caused by parallax, and as the distance $EE'$ is about 190,000,000 miles, we should

* See Herschel's *Outlines of Astronomy*, p. 288.

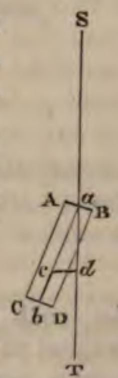
expect that this angle would be sensible: but so enormous is the distance of the fixed stars that ordinary observations detect no annual parallax. By observations of extreme delicacy, Bessel, the late eminent astronomer of Königsburg, detected parallax in one star (61 Cycni); this is a double star, that is, when seen through a good instrument it appears to consist of two distinct bodies revolving about each other, or rather about the centre of gravity of the two, and this was one reason for its selection for the purpose of attempting to detect parallax, because the distance of the point halfway between the two stars from any given point admits of more exact determination than that of the distance of a star itself: another reason for choosing this star was, that it has a larger amount of *proper motion*, that is to say, its place in the heavens instead of being absolutely fixed, is, after every correction has been made, found to vary regularly from year to year, and to whatever cause this may be due, a large amount of proper motion would seem almost certainly to indicate proportionate proximity to our system. The mode of observation was to determine the distance of the point midway between the stars from two other small stars, which it was considered might be supposed to be free from sensible parallax, and by continuing the observations for a year, it was found that after every allowance had been made, there appeared to be a change of position due to annual parallax. The result is, that the star 61 Cycni appears to have an annual parallax of about 00″.348, which gives its distance from the sun to be 657,700 times that of the earth from the sun, or 62,484,500,000,000 miles: light takes rather more than 10 years to traverse this space; yet this is probably one of the nearer of the fixed stars. The period of revolution of the component stars about their centre of gravity is about 540 years; hence, concluding their distance from each other from the preceding calculated distance from the earth and their apparent angular distance from each other, we can deduce the sum of their masses, which proves to be about half that of the sun*.

* There are eight other stars to which annual parallax has been assigned with more or less accuracy. See Herschel's *Outlines of Astronomy*, page 551.

## ON ABERRATION.

46. The third correction to be applied to observations is due to the fact of light travelling with a finite velocity; the manner in which this circumstance affects the apparent position of a star may be explained as follows.

Suppose a particle to move with a uniform velocity from $S$ towards $T$, and suppose it is required to hold a tube $ABCD$, which is also in uniform motion in a direction perpendicular to $ST$, in such a position that the particle shall move along its axis. Let $a$, a point in the axis, be the place of the particle at any given time; let $c$ be any other point; draw $cd$ perpendicular to $ST$, then if the inclination of the axis of the tube to $ST$ be such that




$cd : ad ::$ velocity of tube : velocity of particle,

it is manifest that when the particle has arrived at $d$, $c$ will also have arrived there, and the particle will always be on the axis of the tube, as was required.

Let $\theta$ be the angle of inclination of the tube, $v$ the velocity of the tube, $V$ that of the particle, then we must have

$$\tan\theta = \frac{v}{V}.$$

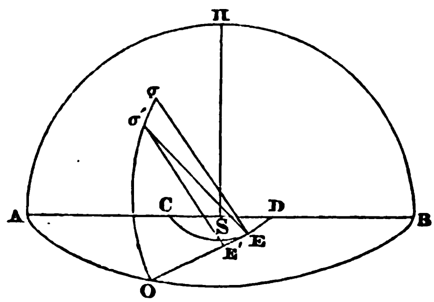
Now the case we have supposed is nearly analogous to that of a heavenly body viewed through a telescope; for the telescope is in motion in consequence of the motion of the point of the earth's surface on which it is fixed, and in order that a star may be visible through it, we must hold it in such a direction that a ray of light coming from the star may pass down its axis; hence we conclude from what precedes, that the direction of the axis of the telescope does not coincide with the direction in which light proceeds from the star, but is inclined at an angle ($\theta$) determined by the equation

$$\tan\theta = \frac{v}{V},$$

where $v$ is the velocity of the earth in its orbit, and $V$ the velocity of light. If then $v$ be comparable with $V$, there will be a consequent error in our observations, and this error is

said to be due to *aberration**. We shall shew how it can be ascertained by observation that the error is appreciable, but first we must explain more particularly the effect of aberration in altering a star's apparent place on the celestial sphere.

47. Let $AOB$ be the ecliptic, $\Pi$ its pole, $S$ the sun, $E$




the earth in its orbit $CED$, $\sigma$ a fixed star. Draw $EO$ a tangent to the earth's orbit, then $EO$ is the direction of the earth's motion, and $\sigma E$ that of light from the star, hence

* The following passage is from Airy's *Ipswich Lectures*: "It was long ago made out that vision is produced by something coming from the object to the eye, and that this something comes from the object to the eye with a definite velocity. Now, in consequence of this light coming from the object to the eye with a definite velocity, and in consequence of the earth's moving with a definite velocity, by the combination of these two things, there is produced a disturbance in the visible place of every object, not connected with the earth, which we look at. Perhaps, one of the simplest ways of giving an idea of the effect of this combination, in relation to the aberration of light, will be to refer you to the chance experiment which suggested the theory of aberration to Dr Bradley, by whom in fact the aberration of light was discovered and reduced to law. He says, he was being rowed on the Thames in a boat, which had a small mast with a vane at the top. At one time the boat was stationary, and he observed by the position of the vane the direction in which the wind was blowing. The men commenced pulling with their oars, and he observed that, at the very time they commenced pulling, the vane changed its position. He asked the watermen what made the vane change its position? The men said they had often observed the same thing before, but did not pretend to explain the cause. Dr Bradley reflected upon it, and was led by it to the theory of the aberration of light. I may offer here a slight illustration of it, which every person may observe if he walks out in a rainy day. If you can choose a day when the drops are large, then if you stand still for a moment, and observe the direction in which the drops are falling, when there is little or no wind, you will see that the drops fall vertically downwards; but if you walk forward, you will see the drops fall as if they were meeting you; and if you walk backward, you will immediately observe the drops of rain falling as if they were coming from behind you. This is an accurate illustration of the principle of the aberration of light." Mr Airy gives the further illustration of a gun fired at a ship in passing a battery; it is evident that if the shot were to enter on one side of the vessel

aberration takes place in the plane $\sigma EO$; join $\sigma O$ by an arc of a great circle, and let $\sigma'$ be the apparent place of the star. Join $\sigma' E$, and draw $\sigma' E'$ parallel to $\sigma E$; then, by what has been said, we must have

$$EE' : \sigma E\,(=\sigma' E) :: \text{velocity of earth} : \text{velocity of light};$$

$$\therefore\ \sin\sigma\sigma' = \sin\sigma E\sigma' = \sin E\sigma' E' = \frac{EE'}{\sigma' E}\sin\sigma' EO,$$

$$\text{or } \sigma\sigma' = \frac{\text{velocity of earth}}{\text{velocity of light}}\sin\sigma O, \text{ nearly.}$$

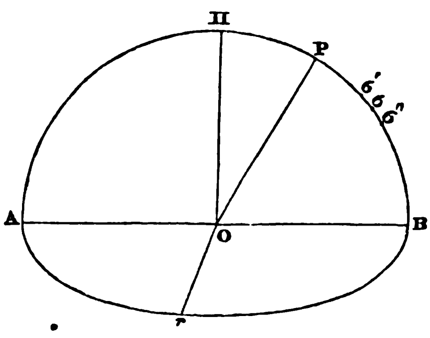
The angle $\sigma O$ is called the *earth's way*.

The velocity of the earth in its orbit is so much greater than that of any point in its surface due to rotation about the axis, that we may confine our attention to the former as producing the error of aberration.

It will be easily seen, that the effect of aberration is to make the apparent place of a star describe a small curve about its real place in the course of a year.

In considering the effect of aberration upon the position of a planet, it is evident that the displacement will depend upon the *relative motion* of the earth and planet.

48. Let us now examine a particular case: suppose $\sigma$ to be on the solstitial colure $A\Pi PB$, and the earth to be in ♈, as




at the autumnal equinox. Then the earth is moving towards

and pass out at the other, then in consequence of the progression of the ship the point of exit of the shot would be astern of the point at which it entered; and if the crew did not take account of their motion they would imagine that the shot was fired from a battery somewhere ahead.

$A$ parallel to the solstitial colure, and consequently the star $\sigma$ is displaced by aberration into the position $\sigma'$, if

$$\sigma\sigma' = a \sin \sigma A = a \sin \sigma B; \left(\text{where } a = \frac{\text{velocity of earth}}{\text{velocity of light}}\right).$$

Again, at the vernal equinox the apparent place of the star will be $\sigma''$, if

$$\sigma\sigma'' = a \sin \sigma B = \sigma\sigma'.$$

Hence we are furnished with the means of determining the coefficient of aberration ($a$) without previous knowledge of the velocity of light. For let $d\,d'$ be the N.P.D. of the star at the autumnal and vernal equinox respectively, then

$$d' - d = \sigma'\sigma'' = 2\sigma\sigma' = 2\,a \sin \sigma B = 2\,a \cos (d + \omega),$$

($\omega$ being $= \Pi P$, the obliquity of the ecliptic);

$$\therefore\ a = \frac{d' - d}{2 \cos (d + \omega)}.$$

49. Hence, also, we are able to determine the velocity of light, for it may be deduced at once from the value of $a$, if the velocity of the earth in its orbit be known; but this is determined by the distance of the earth from the sun, which is known from the value of the sun's parallax. Thus the phænomenon of aberration enables us to determine the velocity of light, and the coincidence of the value thus obtained with that deduced by a perfectly independent method, which will be mentioned in the sequel, leaves little doubt concerning the truth of the principle of both investigations. The coefficient $a$ has, according to the best authority, $20''.445$ for its value, and this gives $8^{m}17^{s}.78$ as the time required by light to traverse the mean radius of the earth's orbit.

## ON PRECESSION AND NUTATION.

50. The preceding articles contain an account of the three astronomical corrections which are necessary to be applied to any observation; but besides these, another correction, though of a different kind, must be applied to the R.A. and N.P.D. of a star, as determined by observation, in order to

make them coincide with those registered in a catalogue of stars. To understand this, it is only necessary to observe, that the R.A. and N.P.D. of a star, when considered as fixing its position, necessarily assume the pole of the equator and the point ♈ to be fixed with regard to the ecliptic, which they are not. We have already alluded in a cursory manner to this fact; we shall now attempt an exposition of its physical cause.

We shall hereafter shew how to prove by observation a fact which has been already mentioned, and which we shall here assume, namely, that the figure of the earth is not accurately spherical, but that of an oblate spheroid or surface generated by the revolution of an ellipse about its minor axis.

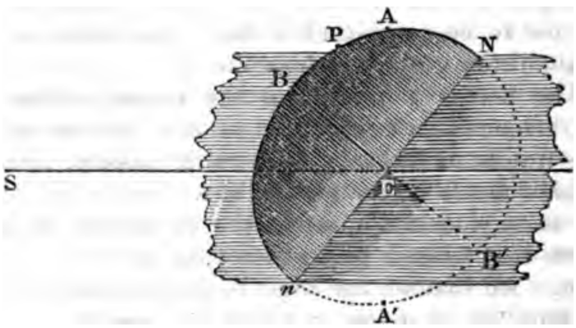
It is to the action of the sun upon the protuberant matter in the neighbourhood of the earth's equator that the phænomenon now to be considered is due; in attempting however to explain the manner in which the result is brought about, it must be borne in mind, that the proper solution of the problem belongs to a department of mathematics with which we have not been hitherto concerned, namely, that department which treats of the motion of a rigid body; and every attempt at explanation must in the absence of an acquaintance with that branch of mathematics be more or less incomplete. All that we can hope to do is to give an imperfect insight into the manner in which the result is produced. The student will find the subject treated at considerable length both in Herschel's *Outlines of Astronomy*, and in Airy's *Ipswich Lectures*.

51. Let us first observe that the force of the sun upon any particle of the earth's mass, with which we are concerned, is not the whole amount of the attraction of the sun upon that particle, but the difference between the attraction upon it and upon a particle at the earth's centre; for we are concerned not with the actual motion of the particles of the earth's mass in space, but with their motion relatively to the centre considered as fixed; now the attraction of the sun upon a particle nearer to it than the earth's centre will be greater

than its attraction upon the earth's centre, and the excess of force will therefore tend from the particle in question *towards* the sun; for like reasons the excess of force upon a particle more distant than the earth's centre will tend *from* the sun. It is manifest also that this excess of force will always be a very small quantity.

This being premised, let us consider the following problem, from the solution of which we shall be able to infer the nature of the motion in the case with which we are actually concerned. A particle $P$ revolves uniformly in a circle about a centre of force $E$, and is disturbed by a very distant body $S$ not in the plane of its motion; to determine the effect of the disturbance upon the motion of $P$.

Let $S$ and $E$ lie in the plane of the paper, and let $NPn$ represent the orbit of $P$; $Nn$ being the intersection of the




plane of the orbit with a plane which for distinctness' sake we will consider to be the plane of the paper, or the line of *nodes*. And suppose the motion of $P$ to be opposite to that of the hands of a watch, so that $N$ is the *ascending* node and $n$ the *descending*.

Let $A$, $A'$ be the points in the orbit which are at the same distance from $S$ as $E$, that is, on account of the great distance of $S$, the points in which a plane through $E$ perpendicular to $SE$ cuts the orbit; and let $B$, $B'$ be the points halfway between $N$ and $n$. Then by what has been already said, through the portion $ABA'$ of the orbit the force upon $P$ is *towards* $S$, and through $A'B'A$ it is *from* $S$. Moreover from $N$ to $B$ and from $n$ to $B'$ the body $P$ is receding from

the plane of the paper, and from $B$ to $n$ and from $B'$ to $N$ it is approaching it.

Now if while $P$ is moving *from* the paper between $A$ and $B$ it receives a small impulse tending *towards* $S$, it is easy to see that the effect will be to make it proceed in a direction less inclined to the plane of the paper, and the direction of its motion produced will therefore cut the plane of the paper to the right of $N$; if therefore after this impulse there should be no further action of $S$, the body $P$ would continue to revolve in a plane *less inclined* to the plane of the paper than $NBn$, and the line of whose nodes would be slightly inclined to $Nn$ and turned in the direction contrary to that of $P$'s motion; this latter change we express by saying, that the line of nodes has *regreded*. This being the effect of a small instantaneous force, we may conclude that the effect of the continuous force of $S$ acting upon $P$ during its passage from $A$ to $B$ is to make the inclination of the orbit continually *diminish*, and the line of nodes *regrede*.

In like manner it will be seen that from $B$ to $n$, when the body is moving *towards* the plane of the paper and the force is *towards* $S$, the effect will be to make the inclination *increase*, and the line of nodes *regrede*.

And from $n$ to $A'$ the effect will be to make the inclination *increase*, but the line of nodes *progrede*.

If then we denote the angle $NAE$ by $\phi$, we find that in passing from $A$ to $A'$, the inclination increases through an angle $90^0 + \phi$ and diminishes through $90^0 - \phi$; and that the node progredes through $\phi$ and regredes through $180^0 - \phi$.

Similar effects take place in the other half of the orbit; hence in an entire revolution the inclination increases through an angle $180^0 + 2\phi$ and diminishes through $180^0 - 2\phi$; and the node progredes through $2\phi$ and regredes through $360^0 - 2\phi$.

Hitherto we have supposed the line of nodes to lie in what would be called in Trigonometry the first and third quadrants; if we had supposed them to lie in the second and fourth, we should have found that the inclination would have increased through $180^0 - 2\phi$, and diminished through $180^0 + 2\phi$; and that the node would have progreded through $2\phi$ as before, and regreded through $360^0 - 2\phi$.

Combining then the results of these two cases we may conclude, that the inclination of the orbit will not be permanently affected, but that on the whole the node will *regrede*.

The preceding investigation is worthy of the student's attention* independently of our present purpose, since it corresponds to the actual case of the moon's motion as disturbed by the sun; we now proceed to deduce from it an explanation of the phænomenon of precession.

Suppose in the first instance that the earth is spherical, then the attraction of the sun upon it will have no tendency whatever to alter its motion with regard to its own centre. But suppose that there is a particle revolving round the earth, and having for the plane of its orbit the plane of the equator; then we have seen that the effect of the sun upon such a particle is not on the whole to alter the inclination of the plane of its motion to the plane of the ecliptic, but to make its line of nodes regrede. Hence then if we suppose a number of such particles to be affixed to the earth at its equator, we may conclude that although at any given moment the effect upon the earth's motion resulting from its attachment will be different for different particles, yet *on the whole* the effect will be to give to the line of equinoxes a motion of regression, and to leave the obliquity of the ecliptic unchanged. And from this it may be concluded, that the effect of the sun's attraction upon the whole of the protuberant matter in the equatorial regions will be of the same kind; the whole result being of course small, on account of the comparatively small quantity of matter upon which the sun acts, and the enormous inertia of the spherical portion of the earth to which this matter is attached.

This mode of considering the subject, though (as has been mentioned) extremely imperfect as a complete solution of the problem, derives considerable interest from the fact of its being the mode in which Newton actually treated the question, when he gave the first explanation of the phænomenon of Precession. Having treated of the motion of the nodes of the moon's orbit, he then considers the effect of a number

* The student's attention may be directed with advantage to Airy's *Gravitation*, where he will find this kind of reasoning extensively used.

of moons attached to the body of the earth. "Par est ratio nodorum annuli lunarum terram ambientis; sive lunæ illæ se mutuo non contingant, sive liquescant et in annulum continuum formentur, sive denique annulus ille rigescat et inflexibilis reddatur." *Princip.* Lib. III. Prop. XXXIX.

52. The general result then of the sun's attraction on the protuberant matter, in the mid-regions of the earth's surface, is to make the first point of Aries *regrede*, that is, move in a direction opposite to that of the earth's rotation, or of the sun's motion, without affecting the obliquity of the ecliptic. This regression of ♈ gives rise to what is called the *precession of the equinoxes;* for since ♈ moves in the opposite direction to the sun, it goes as it were to meet the sun, and consequently the time of arriving at the equinox will precede the time at which the sun would reach it if it were stationary. The whole phænomenon of the motion of the first point of Aries is called *Precession;* the motion is about $50''$ annually. It is not difficult to see, that in consequence of the regression of the line of intersection of the equator and ecliptic, the pole of the former will describe about that of the latter a small circle of the celestial sphere; in other words, the straight line drawn through the earth's centre, perpendicular to the plane of the equator, will describe about the line perpendicular to the plane of the ecliptic, a cone having for its semi-vertical angle the obliquity of the ecliptic.

The results of this precessional motion are remarkable: it will be easily seen that the fixed stars will change their position with respect to the equator and its pole, and the change of their position will be represented by supposing the whole celestial sphere to revolve with a slow angular motion from west to east, about an axis passing through the pole of the ecliptic; the time of this revolution is about 26,000 years. As an illustration it may be observed, that it is only an accident of the present age, that we have what is called a pole star; the star which is now so near the pole, will, after a series of years, leave that position, and only return to it after a complete revolution of the heavens.

It may also be observed, that when the names, Aries,

Taurus, &c., were given to the signs of the zodiac, the beginnings of those signs were found in constellations bearing those names; but now the *signs* are far distant from the *constellations* which have respectively the same names.

53. It is not difficult to see, that the action of the sun on the earth is not exactly the same for all periods throughout the year; the mean effect on the pole of the equator will be such as has been described, but to represent the facts more accurately, we must suppose the pole to describe about the pole of the ecliptic, not a small circle, but a tortuous curve, sometimes approaching the pole of the ecliptic, sometimes receding from it, this curve, however, never differing much from a circle. This irregularity of the motion of the pole is called its *nutation*.

54. What has been said hitherto has been confined entirely to the action of the sun upon the earth in producing precession and nutation; a similar effect will manifestly be due to the action of the moon, and the entire result will be due to the joint action of the two bodies. Although the mass of the moon is so much smaller than that of the sun, yet on account of her proximity the precessional motion due to her action is greater than that due to the sun in the proportion of about 5 to 2. The nature of the moon's action moreover is of a much more complicated kind than that of the sun, on account of her orbit not coinciding with the plane of the ecliptic, nor even having a fixed position with respect to that plane on account of the regression of the nodes of her orbit before referred to; the portion of the nutation therefore due to the moon will depend partly upon the moon's position in her orbit, and partly upon the position of the line of nodes; the general nature however of the mechanical action is the same for both sun and moon, and the explanation which has been given for one will apply in principle to the other.

55. We shall conclude this account of the corrections which must be applied to astronomical observations, by giving an account of the manner in which Bradley separated the

effects of Aberration and Nutation and established the existence of both.

The star $\gamma$ *Draconis*, passing near the zenith of Bradley's observatory, and being consequently little affected by refraction, was the chief star of his observations. This star in March passed more to the south of the zenith by about 39″ than it did in September. Other stars also changed their declinations; a small star in *Camelopardalus*, having an opposite R.A. to that of $\gamma$ *Draconis* was observed at the same time; and Bradley argued that if the changes of declination arose from a real nutation of the earth's axis, the pole must have moved as much towards $\gamma$ *Draconis* as from the star in *Camelopardalus;* such however was not the fact, and Bradley was led to account for the phænomena by the theory of aberration. Now if this theory, together with that of precession, would account for all observed changes of declination, the declination of $\gamma$ *Draconis* in September 1728 ought to have differed from that in September 1729 merely by the quantity due to precession, since the aberration is the same at the same season of the year. This proving not to be the case, it appeared that there was still some cause of change of declination undiscovered.

Bradley was thus led to readopt the hypothesis of a nutation of the earth's axis, which he had before rejected; he found that within the same periods the changes in declination of $\gamma$ *Draconis* and of the star in *Camelopardalus* were equal and in contrary directions, and these changes which were observed through a term of years could be satisfactorily explained by supposing a nutation of the earth's axis from the one star and towards the other.

After 1731 Bradley observed contrary effects to happen; that is, $\gamma$ *Draconis* receded from the zenith and north pole, and the star in *Camelopardalus* by equal steps approached those points; and this state of things continued for more than nine years, from 1731 to 1741, after which $\gamma$ *Draconis* again began to approach the zenith and the other star to recede. These phænomena then might be explained, by supposing that from 1731 to 1741, there was a nutation of the earth's axis from $\gamma$ *Draconis* and towards the star in *Camelopardalus*.

Bradley was thus led to connect the physical cause of the phænomenon with the position of the moon's orbit, the nodal line of which performs a complete revolution in about 19 years, and to refer the supposed nutation of the earth's axis to the variable effect of the moon in producing precession. Subsequent examination has left no doubt of the truth of this peculiarly ingenious hypothesis; indeed the whole question of precession and nutation has been reduced to the most accurate mathematical calculation upon the principles of universal gravitation. "We cannot," says an eminent scientific writer, "sufficiently admire the patience, the sagacity, and the genius of the Astronomer, who, from a previously unobserved variation not amounting to more than forty seconds, extricated, and reduced to form and regularity, two curious and beautiful theories*."

## ON THE PROPER MOTION OF THE FIXED STARS.

56. If the R.A. and N.P.D. of any fixed star be found by observation, and all the corrections which we have described be applied, it will nevertheless be found that the position of the star is not actually fixed. The change of position is very small and regular, and for many stars is a known quantity. No physical cause has been certainly assigned, but the variation is attributed to a proper motion of the stars themselves. And reflection will shew that such a proper motion might have been expected *à priori;* for supposing the law of attraction which holds in our own system to extend, as most probably it does, to the regions of the fixed stars, then the mutual attractions of the stars must prevent them from being actually fixed in space. Nevertheless, in consequence of their enormous distance, the proper motion of a star will produce an extremely small apparent change of position to an observer on the earth's surface, and therefore we still use the name *fixed* stars, although believing the bodies so called to be really in motion.

Regarding the sun as one amongst the stars we should judge from analogy that it was not absolutely fixed in space, but that it had a proper motion of its own. And this appears

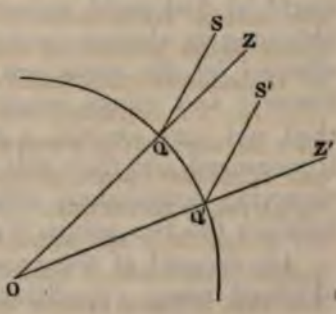
* Woodhouse's *Astronomy*, from which the previous account is taken.

to be the fact, and it is a subject of much interest to determine the point of the heavens to which the solar system is moving, and if possible also the velocity of its motion. The solution of the problem involves many sources of uncertainty; nevertheless there is so near an agreement in the results of independent investigators, as to make it highly probable that the point towards which the sun's motion was taking place at the epoch 1790 was nearly that determined by R.A. $260^0$ and N.P.D. $55^0$. With regard to the velocity there is more uncertainty, but it is probable that the rate of motion is about 422,000 miles per diem, or that the velocity is somewhat more than one-fourth of the velocity of the earth in her orbit. On this curious subject we must refer to Herschel's *Outlines of Astronomy*, Chap. XVI. where the student will find a variety of information respecting the fixed stars, which it would be impossible to introduce into a short treatise.

## ON THE MODE OF DETERMINING THE EARTH'S FIGURE AND MAGNITUDE.

57. We have already described the earth as being nearly spherical, but as being more accurately of the form of an oblate spheroid, or of the figure which would be generated by the revolution about its minor axis of an ellipse of very small excentricity. We shall now endeavour to explain in a general way the method of determining by observation and calculation the magnitude and figure of the earth.

Let us first adopt the hypothesis that the earth is spherical, and proceed upon that hypothesis to determine its radius. Let $O$ be the earth's centre; $Q$, $Q'$ two places upon the same meridian, the distance between which is accurately measured; also let the meridian zenith distance of the same star $S$ be observed at $Q$ and $Q'$; then the angle $QOQ'$ will evidently be equal to the difference between these two zenith distances, and hence the angle $QOQ'$ is known



from astronomical observation; but if we know $QQ'$ and the angle $QOQ'$, we shall know the radius $OQ$ or $OQ'$; and hence the radius of the earth supposed spherical may be found.

If however the radius were so determined by means of observations made at a number of different places upon the earth's surface, it would be found, that although in all cases we should obtain a result not differing much from 4000 miles, yet there would be discrepancies sufficiently large to shew, that the hypothesis of the earth being spherical is untenable as an accurate representation of the fact, and that it is only approximately correct. What we do in reality determine by such a process as that above described, is the radius of curvature of the earth's surface, or its curvature in the neighbourhood of the places of observation. In the figure $ZQO$ will be the vertical line, or the direction of the plumbline at the place $Q$, and $Z'Q'O$ will be the corresponding direction at $Q'$, but $O$ will not be the centre of the earth; the angle $QOQ'$ will however be determined as before, being in fact the difference of latitude between the two places. The curvature of the earth's surface having been thus determined for a number of different places, it remains to determine the form which will best satisfy all the various results.

The difficulty of determining the earth's figure is thus reduced to that of ascertaining the length of the arc $QQ'$; this we have spoken of as being accurately measured; in reality however the length must be ascertained by a trigonometrical survey, in the conduct of which the utmost care is required. In the first place a base line must be actually measured; and this is a work by no means so simple as might at first sight appear, on account of the difficulty of obtaining an absolute measure of length which shall be unaffected by the variations of temperature. This difficulty however and others being overcome by methods which we shall not here explain, a base line in a convenient position is measured, and from this a system of triangles is formed by observing conspicuous objects, until at length the distance between the two points $Q$ and $Q'$ is determined.

On account of the great importance of the result to be obtained many expeditions have been fitted out, for the

purpose of measuring an arc of the meridian at different parts of the earth's surface. From a comparison of all the measures Mr Airy has arrived at the following results, that the earth may be regarded as an oblate spheroid, of which

$$\text{the major axis} = 7925.648 \text{ miles}$$
$$\text{and the minor axis} = 7899.170 \text{ miles.}$$

Hence the polar diameter is less than the equatorial by about $\frac{1}{300}$th part of the whole; this fraction then measures what is technically called the *compression*.

58. It can scarcely be said that this result requires confirmation, but it is satisfactory to know that the form of the earth here assigned agrees with the results made by means of the pendulum upon the intensity of the earth's attraction in different latitudes. Putting out of consideration the earth's form, and supposing it truly spherical, there would still be a variation in the intensity of gravity in different latitudes; for gravity does not arise entirely from the earth's attraction, but is the resultant of the earth's attraction and the centrifugal force due to the earth's rotation; and, since particles in different latitudes are at different distances from the axis of rotation, the amount of centrifugal force is different, and therefore the intensity of the resultant force upon them. Now the intensity of gravity in different latitudes can be ascertained with great accuracy, by means of observations of the pendulum; and the degree in which that force ought to vary in consequence of centrifugal force only, can also be determined; and when we compare the two results we find that they do not agree, thus proving that a cause of variation in the force of gravity exists besides that arising from centrifugal force; this discrepancy is satisfactorily disposed of by consideration of the earth's spheroidal form.

There is still another and independent method of confirming the above view of the earth's form, which is too curious to be omitted. The motion of the moon with respect to the earth is different upon the hypothesis of the earth being a spheroid, from what it would be if the earth were a sphere. Two irregularities in the moon's motion Laplace

was led to assign to this cause; from one of which he deduced a compression of $\frac{1}{305.05}$, and from the other a compression of $\frac{1}{304.6}$.

Lastly it may be noticed, that the problem of the figure of the earth has been treated as a purely hydrostatical problem, upon the hypothesis of its original fluidity. It is evident that if a spherical mass of fluid be made to revolve slowly about an axis, the effect of centrifugal force will be to cause a protuberance in the equatorial regions; great difficulty is however introduced into the problem, by an ignorance of the manner in which the density of the fluid of which the earth is supposed to be composed varies in the interior, and results obtained upon any arbitrary hypothesis concerning this change of density cannot be regarded as otherwise than very uncertain. In this manner the compression of the earth has been found to be $\frac{1}{307.313}$, a result which certainly agrees with observation in a very remarkable manner, but which is nevertheless of no great value for the reason above mentioned.

## ON THE PLANETS.

59. We shall now proceed to give a more accurate account of the sun and the bodies revolving about it, of which the earth is one.

The principal planets are named as follows, the order being that of their distances from the sun, Neptune, Uranus, Saturn, Jupiter, Mars, the Earth, Venus, Mercury. Besides which there are no less than forty-four very small planets, called *asteroids;* these are invisible to the naked eye, and in point of distance from the sun are intermediate to Mars and Jupiter. Even now others may exist which have not been recognised; for among the multitude of telescopic stars, few, comparatively speaking, have been sufficiently observed to enable us to decide upon the constancy of their position. No less than twenty have been added to the list of asteroids since the publication of the last edition of this work.

The following Table exhibits the mean distances of the principal Planets from the Sun*.

| | |
|---|---|
| Mercury | 0.387 |
| Venus | 0.723 |
| Earth | 1. |
| Mars | 1.524 |
| Jupiter | 5.203 |
| Saturn | 9.539 |
| Uranus | 19.182 |
| Neptune | 30.037 |

And the following is a list of the asteroids with their respective mean distances from the sun and the dates of discovery†; the list does not include one discovered by Mr Pogson at the Oxford Observatory in April of the present year, (1857), which has been named Ariadne, nor one discovered in last May at Paris by M. Goldschmidt, which has not yet received a name.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | Ceres | 2.767 | 1801 | 22 | Calliope | 2.909 | 1852 |
| 2 | Pallas | 2.770 | 1802 | 23 | Thalia | 2.626 | 1852 |
| 3 | Juno | 2.669 | 1804 | 24 | Themis | 3.141 | 1853 |
| 4 | Vesta | 2.361 | 1807 | 25 | Phocæa | 2.401 | 1853 |
| 5 | Astræa | 2.576 | 1845 | 26 | Proserpina | 2.655 | 1853 |
| 6 | Hebe | 2.425 | 1847 | 27 | Euterpe | 2.346 | 1853 |
| 7 | Iris | 2.387 | 1847 | 28 | Bellona | 2.775 | 1854 |
| 8 | Flora | 2.201 | 1847 | 29 | Amphitrite | 2.545 | 1854 |
| 9 | Metis | 2.385 | 1848 | 30 | Urania | 2.364 | 1854 |
| 10 | Hygea | 3.149 | 1849 | 31 | Euphrosyne | 3.156 | 1854 |
| 11 | Parthenope | 2.452 | 1850 | 32 | Pomona | 2.583 | 1854 |
| 12 | Victoria | 2.335 | 1850 | 33 | Polyhymnia | 2.866 | 1854 |
| 13 | Egeria | 2.576 | 1850 | 34 | Circe | 2.667 | 1854 |
| 14 | Irene | 2.585 | 1851 | 35 | Leucothea | 2.974 | 1855 |
| 15 | Eunomia | 2.643 | 1851 | 36 | Atalanta | 2.757 | 1855 |
| 16 | Psyche | 2.923 | 1852 | 37 | Fides | 2.654 | 1855 |
| 17 | Thetis | 2.473 | 1852 | 38 | Leda | 2.635 | 1856 |
| 18 | Melpomene | 2.296 | 1852 | 39 | Lætitia | 2.765 | 1856 |
| 19 | Fortuna | 2.443 | 1852 | 40 | Harmonia | 2.268 | 1856 |
| 20 | Massalia | 2.409 | 1852 | 41 | Daphne | (not known) | 1856 |
| 21 | Lutetia | 2.435 | 1852 | 42 | Isis | 2.290 | 1856 |

* Taken from Herschel's *Outlines of Astronomy*.

† Taken from a Tract entitled "De planetis minoribus inter Martem et Jovem circa Solem versantibus, Dissertatio Astronomica inauguralis, auctore C. C. Bruhns."

60. The paths of the planets, with the exception of the asteroids, are in planes making a small angle with the plane of the ecliptic; their motions are governed by the three following remarkable laws, called from their discoverer Kepler's Laws*.

I. *The planets move in ellipses, each having the sun's centre in one of its foci.*

II. *The areas swept out by each planet about the sun are, in the same orbit, proportional to the time of describing them.*

III. *The squares of the periodic times are proportional to the cubes of the major axes.*

It will be easily seen that these three laws follow at once from the hypothesis of a force, varying inversely as the square of the distance, resident in the sun's centre. (See *Newton*, Props. I. XI. and XV.) Speaking accurately, the centre of the sun is not a fixed point, the motion taking place in fact about the centre of gravity of the whole system; but on account of the enormous mass of the sun, as compared with any one or with all of the planets, the centre of gravity is very near the centre of the sun, and may generally be conceived of as coincident with it. Moreover, no one of Kepler's laws is quite rigidly true, because not only does the sun attract each of the planets, but the planets mutually attract each other, and produce slight perturbations or deviations from laws which would hold strictly for any one undisturbed planet.

The complete investigation of the motion of a disturbed planet requires the most refined mathematical processes, and can only be understood by those who have made themselves masters of the methods of modern analysis. Sir John Herschel has, in his *Outlines of Astronomy*, succeeded in giving a more popular view of the subject; nevertheless even in this form much study will be required in order to obtain any valuable insight into the nature of the solution of the problem.

* The history of Kepler's discoveries, which may be found in his life published by the Society for the Diffusion of Useful Knowledge, is well worthy of the student's attention.

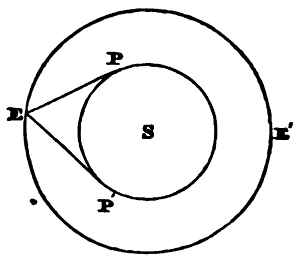
Perhaps the most interesting result of the mathematical investigation is that, which assures us that the orbits of the planets cannot change materially either in magnitude or in form or in relative inclinations from what they are at present; so that, as far as existing causes are concerned, we are assured of the stability of the system, notwithstanding all mutual perturbations of the constituent bodies.

Nothing can give a more convincing proof of the truth of the great principle of universal gravitation, of which the theory of the planets is perhaps the most striking application, than the actual solution of an inverse problem of planetary perturbation resulting in the discovery of the planet Neptune. In the direct problem, we know the position of the disturbing bodies, and the thing to be done is to determine the motion of any one planet as affected by the action of the rest; but the following problem is conceivable,—Given the perturbations of a planet to determine the magnitude and position of a body which will produce them; and this problem has been in one instance solved. The motion of the planet Uranus was known to be disturbed in a manner which could not be, or at least had not been, explained by the action of the recognised members of the Solar system; it was suggested, that the perturbations might be due to the action of a planet whose orbit was exterior to that of Uranus, and on this supposition two mathematicians, Mr Adams in this country and M. Leverrier in France, independently undertook, and both with success, the search for the hypothetical planet. Such is a brief sketch of the discovery of Neptune, which may justly be ranked as among the most remarkable achievements of modern science.

61. The planets which are at a greater distance from the sun than the earth, are called *superior* planets, those which are at a smaller distance, *inferior*. It will be anticipated that the phænomena exhibited by these two classes to an observer on the earth's surface will be in many respects different.

Let $S$ be the sun's centre, $EE'$ the earth's orbit, $PP'$ the orbit of an inferior planet, supposed for simplicity's sake to be in the plane of the ecliptic. Then it is evident that the planet's *elongation* from the sun, that is, the angle subtended

at the earth by its distance from the sun, can never exceed a certain limit; for from $E$, the position of the earth, draw $EP$, $EP'$ tangents to the planet's orbit, then $PES$, or $P'ES$, will be the maximum angle of elongation, and the elongation will vary between zero and this value; sometimes the planet may even pass between the earth and the sun, and it will then be seen to cross over the sun's face like a dark spot. Now suppose the




sun and the planet to revolve round $E$, then the time of passing the meridian of any given place will not be very different for the sun and the planet; for example, let Venus have her maximum elongation, and suppose her to rise some time before the sun so as to be a *morning* star, then the time by which her rising precedes that of the sun diminishes until at last she is so near the sun that she becomes invisible: afterwards she begins to rise after the sun, and therefore to set after him, and thus becomes an *evening* star. Mercury is seldom sufficiently far from the sun to be visible with the naked eye. There is manifestly no limit to the elongation of the superior planets, and therefore no connexion between the times of their rising or setting and that of the sun.

The planets circulate round the sun in the same direction, and it may be mentioned that the stability of the system alluded to in the preceding article is partly due to this cause; if, however, the places of the planets be observed from the earth and referred to the celestial sphere, their motion will be of a complicated kind, that is, their motion will sometimes appear direct or in the order of the signs of the zodiac, as they would if seen from the sun, sometimes the motion will appear retrogade or contrary to the order of the signs, and sometimes they will appear to move neither one way nor the other, but to be at rest. The student will understand that we are not here speaking of the apparent diurnal motion due to the rotation of the earth, but to the course which would be represented if the place of a planet were noted every night and marked down upon a globe representing the celestial sphere. With regard to the inferior planets this result

may be deduced as follows: when Venus (for instance) is on the opposite side of the sun from the earth, or in heliocentric opposition, she will appear to be moving in the order of the signs as if seen from the sun; but if she is in heliocentric conjunction, she will appear from the earth (supposed for the moment stationary) to be moving in the opposite direction from that in which she moved before; the earth however is not stationary, but her angular motion is less rapid than that of Venus by Kepler's third Law, hence the motion of Venus will still appear to retrograde. Between these two opposite states of motion there will be an epoch at which the one will change to the other, and for which therefore the planet will appear stationary.

The same thing holds true of a superior planet. Suppose that Jupiter and the earth are in heliocentric opposition, then the motion will appear direct, exactly as if he were an inferior planet. Again, suppose them in heliocentric conjunction, then if the earth were stationary, Jupiter would appear to be moving in the order of the signs; but the earth is moving in that direction also and more rapidly than Jupiter, hence, *relatively*, Jupiter moves backward or contrary to the order of the signs. And it necessarily follows that between these the two opposite kinds of motion there must be stationary points.

This kind of reasoning then demonstrates that all planets, whether inferior or superior, must alternately be direct and retrograde in their motion as seen from the earth.

62. The inferior planets present to the earth *phases*, exactly as does the moon: this will be seen to be a necessary result of the manner in which they are illuminated, for since only one hemisphere of the planet, namely that which is turned towards the sun is illuminated, the portion of the illuminated surface visible from the earth must depend upon the relative positions of the sun, earth, and planet. Of the superior planets, Mars, which is the nearest to the earth, presents sometimes a slightly *gibbous* appearance, that is, a phase like that of the moon when near full, but the others have no perceptible changes of phase, because the direction in which light falls

upon them from the sun is, on account of their great distance, very nearly the same as that in which they are viewed from the earth.

63. The limits of this treatise will not allow us to enter upon a description of the physical peculiarities of the various planets. We cannot however omit to take notice of the very remarkable object called Saturn's *ring*. It would be more correct to speak of two rings, for there are in fact two, lying in one plane and separated from each other by a narrow interval, the inner one being separated by a much wider interval from the planet itself*. The ring has a rapid rotation in its own plane, the centrifugal force arising from which is doubtless the means of preserving its form and its position.

64. The earth and the superior planets are accompanied by secondary bodies, termed *satellites*, revolving about them in the same manner as they themselves revolve about the sun. Of these satellites or moons, the earth has one, Jupiter four, Saturn seven, Uranus certainly two and perhaps six; only one satellite has at present been observed as attending Neptune.

For much interesting information on this subject, we must again refer to Sir J. F. W. Herschel's *Outlines of Astronomy*, a work to which reference has been frequently made already, and the excellence of which it is impossible to overrate.

## ON THE MOON.

65. The earth is accompanied (as has been already mentioned) in its course round the sun by a secondary body, which we call the moon, and which revolves round it in the same kind of way as the earth revolves about the sun, and from west to east. The distance of the moon from the earth, as concluded from the value of its horizontal parallax, is about 60 of the earth's radii. The moon revolves round the earth according to the ordinary rules of bodies moving in central

* Recent observations appear to have demonstrated the existence of another ring, lying between the ring as formerly observed and the body of the planet. This new ring can only be seen under most favourable circumstances and by a most practised observer.

orbits, deviating however from the strictness of these rules materially in consequence of the disturbing force of the sun, which is much greater than that of one planet on another, and therefore causes the moon's orbit to deviate more from the elliptical form than is the case with the orbit of a planet. The time of the moon's revolution about the earth is 27 days 7 hours nearly.

Some of the deviations of the moon from uniform circular motion are so great as to have been noticed by very early astronomers; and these were known as facts of observation long before their physical cause was assigned. Theory has added many more errors, or deviations from undisturbed motion, to the list, and the theory of the moon's motion may now be regarded as complete. It may be remarked of the moon, as of the planets, that the complete investigation of the orbit requires the most refined mathematical analysis; but the principal irregularities may be explained without difficulty by general reasoning upon the nature of the disturbing force of the sun, and the student will find the subject thus treated with great clearness in Airy's *Gravitation;* the regression of the nodes of the moon's orbit has already been incidentally introduced into this treatise. Newton himself applied his theory of universal gravitation to the motion of the moon, and with great success; he was however able to calculate mathematically only the principal errors.

66. The most remarkable phænomenon exhibited by the moon is the change of its *phase*, which has been already alluded to in speaking of the phases of the inferior planets, and which is an immediate result of being illuminated by the sun. It is manifest that when the moon and sun are on opposite sides of the earth, or the moon is in *geocentric opposition*, the bright side of the moon will be turned towards the earth, or it will be *full-moon;* and when the moon is between the earth and sun, or in *geocentric conjunction*, the bright side will be turned away from the earth, or it will be *new-moon*. Between these two extreme cases the moon will present every variety of phase. The time elapsing between two successive full-moons is not that of the moon's revolution, because

the sun has, during the month, advanced in its course; the time will be equal to the time of revolution of the moon, together with that which the moon takes to pass over the space moved through by the sun, or about 29½ days. This is called a *lunation*, or a *synodic period*.

67. It may be noticed that the moon when full, being in exactly the opposite quarter of the heavens from the sun, or in other words, distant from the sun 180° in R.A., comes upon the meridian 12 hours after the sun, i.e. at 12 o'clock at night; hence the moon is brightest at a time when the darkness of the night would otherwise be the most intense.

68. The moon revolves on its own axis in the same direction in which it revolves about the earth, and in the same period; the consequence is that it always presents to the earth the same face. The appearance of the face is extremely ragged and mountainous; the height of some of the mountains has been ascertained, and it thus appears that they are very much greater as compared with the size of the moon, than in the case of terrestrial mountains.

It may be remarked, that it is not strictly true that the moon always presents exactly the same face, or that we are able to see just half of the moon's surface and no more; and this for two reasons. In the first place, the rotation of the moon on its axis is uniform, but the moon's motion in its orbit is not so, and the consequence is that we are able to see a little beyond the border of the hemisphere, which would be illuminated if the motion in its orbit were perfectly uniform. And again, since the axis of the moon is not accurately perpendicular to the plane of motion, the two poles with the regions just beyond them come alternately into view. These phænomena are known as the moon's *Librations*.

To a spectator upon the moon's surface the earth must present the appearance of a moon immoveably fixed in the sky, and going through phases in the same manner as the moon does to an inhabitant of the earth.

The moon has no clouds, nor any other decided indication of an atmosphere. Hence the climate of the moon

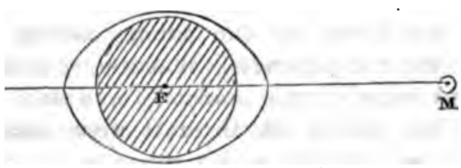
must be most remarkable; the changes being from the brightest sunshine during one fortnight to excessive frost and cold during the next. "Such a disposition of things," says Sir John Herschel, "must produce a constant transfer of whatever moisture may exist on its surface, from the point beneath the sun to that opposite, by distillation *in vacuo* after the manner of the little instrument called a *Cryophorus.* The consequence must be absolute aridity below the vertical sun, constant accretion of hoar frost in the opposite region, and perhaps a narrow zone of running water at the borders of the enlightened hemisphere."

The most obvious office of the moon is that which it performs in giving light to the earth, but an equally important one is that of producing tides, of which we shall presently speak more particularly.

The phænomenon of eclipses also, which the moon is instrumental in producing, will be separately considered.

## ON TIDES.

69. The accurate theory of tides is one of the utmost difficulty; but a general explanation, sufficient for many purposes, may be given as follows.




Let $E$ be the earth, $M$ the moon; and suppose the earth to be surrounded by a sea, and the moon to be at rest with respect to it. Then the attraction of the moon on each particle of water will tend to draw the water away from the earth on the side turned towards it, and thus to make a protuberance of water: there will be a similar protuberance on the opposite side of the earth, because the force which draws any particle of water away from the earth is the difference between the attraction of the moon on that particle and on the earth,

and hence the force in question upon a particle on the opposite side of the earth from the moon will tend from the earth; and hence, as we have said, there will be a protuberance of water on the side of the earth which is turned away from the moon, as well as on the other. The accurate form of the surface of the sea is a spheroid, having its longer axis passing through the moon.

If there were no moon there would be a similar disturbance of the spherical form of the sea by the sun's attraction. In order to determine the result of their combined action, it will be sufficient to observe, that if the sun and moon are on the same side of the earth, or the moon in geocentric conjunction, the action of the two will be combined, and we may suppose the effect to be the sum of those due to their separate actions; the same will be the case when the sun and moon are on opposite sides, or the moon in geocentric opposition: when the moon is halfway between opposition and conjunction, the effect may be considered to be the difference of those due to the lunar and solar action; in other positions of the sun and moon the results will be intermediate. Let us now suppose that as the moon revolves the tidal protuberance follows it, then as the earth turns on its axis each place of the surface will have low and high water succeeding each other at intervals of twelve hours. From what has been said it is clear that the highest tides will be at the new and full moon, and the lowest at the times halfway between; the former are called *spring-tides*, the latter *neap-tides*. In order to make this theory agree tolerably well with observation, it is necessary to suppose that the axis of the tidal spheroid lags somewhat behind the moon.

The preceding account of the tides, which is known as the *Equilibrium Theory*, would be imperfect, even if the earth were entirely surrounded by an ocean in the manner which we have supposed; but the phænomena of tides are still further, and almost indefinitely complicated, by the mixture of land and water upon the earth's surface, so that in particular localities the tidal phænomena differ extremely from that which we have spoken of, namely, high water and low water alternately every twelve hours. The problem of Tides has,

however, been to a considerable extent brought under the dominion of Mathematics, and in Mr Airy's Essay on the subject, in the *Encyclopædia Metropolitana*, many of the most curious phænomena are explained by direct mathematical calculation. We shall conclude this article with an extract on the subject of Tides from Buff's *Physics of the Earth*, a little work which we are glad to take the opportunity of recommending to the student, as containing in a small compass very much interesting information given with much clearness.

"The formation and progress of the Tide-wave, although in general it is connected with the conditions which have been explained, is yet subjected to very considerable changes by the peculiar arrangement of the land and sea. The updrawing of a perfect tide-wave requires that the moon, or the moon and the sun together, should stand in the zenith of some point in the sea, while for two other points, or at least for one, of the same sea they must be just on the horizon. At such latter point the tide is at the lowest, just when it is at the highest at the former. From this it is evident, that neither the inland seas, lakes, nor small seas in general, can be subject to a tide of their own. Even the Atlantic ocean is not broad enough for the formation of a powerful tide-wave. The breadth of this ocean near the equator amounts to forty or fifty degrees, or about one-eighth of the circumference of the earth. But the curvature of the earth's surface is far from being great enough to allow of any considerable difference between the distances of our satellite from any different points of this sea. Only the great Pacific Ocean, whose enormous mass of water embraces nearly half the globe, has width enough for this. The Pacific therefore is the sea from which the tides chiefly come forth. The tide-wave once formed, marches on from this ocean, towards the west, according to the same laws which govern the path of any other wave, which may be raised on any surface of water, whether by the wind, by a stone thrown in, or by any other cause. It reaches the Indian Ocean, partly running round Australia, partly finding its way through the numerous straits of the Indian Archipelago. The Atlantic Ocean, severed almost entirely from the great ocean by the far-stretching

continent of America, receives now most of the remaining force of the wave, as, turning the southern point of Africa, it presses onwards to the North, until it is lost in the Arctic Sea. It is to this that we chiefly owe the tides of our European and American coasts.

"Now the tide-wave requires time for its development, and therefore, at the place of its origin, is not completed till after the moon has passed the meridian. Moreover, in all smaller seas and arms of the sea, in which tides occur, these tides must be due to the progress of the wave derived from the Pacific Ocean; and the obstacles which oppose its advance, and by which it is at last arrested, vary very much according to the form of the coasts, to the width and depth of the sea, and to the number and size of the islands that it meets with. For these reasons the tides that visit the coasts of Europe must be retarded, so as to occur considerably later than the cause from which they arise. Thus, for instance, the tide-wave requires (according to the reckoning of Whewell) fourteen or fifteen hours to travel from the southern end of Africa to the coasts of Spain, of France, and of Ireland. And then, on account of the increased resistance there, seven more hours are necessary for it to get through the English Channel. The North Sea receives its tides from the two branches of the tide-wave, one of which comes through the Channel, while the other passes round Scotland. The same tide which appears at Brest at noon, reaches Dover and Calais about seven o'clock, and Ostend about eight in the evening. The same tide running round Scotland arrives at the mouth of the Thames at eight o'clock on the following morning, as well as on the coast of Germany, when it meets and swells the other wave that came up the Channel.

"The tide-water, rising in front of a river's mouth, partly pours itself into the river, and partly prevents the escape of its waters into the sea, so that, in great rivers, the flood is felt many miles up the stream, being however more and more retarded the further it advances; so much so, that it may easily happen, that at the river's mouth the ebb may have already commenced, while higher up the flood may still be rising.

"The causes, however, of these several delays of the tide remain always the same; the tides must therefore ever follow each other in regular and equal periods. Hence the times of their recurrence may be calculated from the position of the moon any length of time beforehand. The regular delay of high water at any place, after the moon's passage of its meridian, on the days of new and of full moon is called the *Establishment of the Port**."

## ON ECLIPSES.

70. Eclipses are of two kinds, namely, of the moon and of the sun, and may be described in general as being caused respectively, by the passage of the moon through the shadow thrown behind the earth, and by the passage of the moon between the earth and the sun, so as to intercept the sun's light from the earth. It is manifest that if the motions of the sun, earth, and moon, be accurately known, so that their relative positions at any time can be predicted, it will be only a matter of calculation to determine when an eclipse will take place; but the method of calculation must be very different in the case of a lunar from that of a solar eclipse, and the latter will be very much more complicated than the former, as will be seen when we have described the phænomena more particularly.

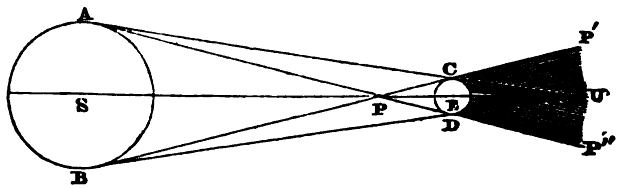
### *Lunar Eclipse.*

71. Let $S$ be the sun, $E$ the earth, draw the common tangents to their surfaces $ACU$, $BDU$, meeting in $U$, and the two $APDP''$, $BPCP'$, meeting in $P$ between the sun and earth.

Then the portion of the cone, of which the vertex is $U$, behind the earth is the earth's shadow, and is called the *umbra;* the portion of the cone, of which the vertex is $P$, behind the earth is partially free from the sun's rays in consequence of the intervention of the earth, and is called the *penumbra.* When the moon is eclipsed it is observed to enter

* Buff's *Physics of the Earth*, p. 28. The student may with great advantage study the chart of the Tides in Johnstone's *Physical Atlas.*

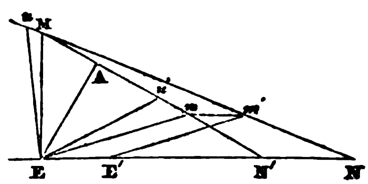
the penumbra first, and afterwards the umbra: on account of the relative magnitude of the earth's diameter and the




distance of the moon, the distance of the vertex of the umbra from the earth is always greater than that of the moon, and hence the moon will always pass through the umbra if at the time of geocentric opposition its path is rightly directed.

72. But there is not necessarily an eclipse of the moon at each opposition, because the moon's orbit is inclined to that of the ecliptic, and at the time of opposition it may not be sufficiently near to a *node*, (or point in which its path crosses the ecliptic,) to pass through the umbra. We shall endeavour to explain the mode of making the calculations necessary to determine whether at any given opposition the moon will be eclipsed, and to what extent.

73. Let $EN$ be a small portion of the path of the centre of the earth's shadow at the distance of the moon, considered as a straight line, $MN$ a portion of the path of the moon's centre, $M$ being the position at the time of opposition, and $N$ the node. Then it will be convenient to suppose the earth to remain fixed, and the moon to move in an imaginary orbit $MN'$,




called a *relative* orbit, such that the distance between the centres of the moon and shadow shall always be the same as they actually are. Let $m'$ be the moon's centre when the centre of the shadow is at $E'$; join $E'm'$, take $m'm$ equal and

parallel to $EE'$, and join $Em$; then $Em$ will be equal and parallel to $E'm'$, and $m$ will therefore be a point in the relative orbit. It is not difficult to see that the relative orbit $MN$ will be a straight line, and its inclination $MN'E$ to the plane of the ecliptic may be calculated from a knowledge of the inclination of the moon's orbit and of the relative velocities of the earth and moon.

The radius of the umbra at the moon's distance, or rather the angle subtended by that radius at the earth, may be easily calculated from the parallax of the sun and moon and their apparent diameters; all which quantities are known and registered.

Now draw $EA$ perpendicular to the relative orbit, then since $ME$, which is the moon's latitude at the time of opposition, and $EMA$ are known, $EA$ may be calculated. $EA$ is the nearest approach of the centres of the moon and the umbra; if then $EA$ be greater than the sum of the radii of the moon and the umbra, there will be no eclipse, if less there will be an eclipse of greater or smaller degree of obscuration according to the value of $EA$. If we draw two lines $Eu$, $Eu'$, each equal to the sum of the radii of the moon and umbra, then $u$ and $u'$ will be the positions of the moon's centre at the commencement and termination of the eclipse respectively; and by calculating these positions we can easily determine the time of the commencement and termination.

74. Since a lunar eclipse is caused by an actual deprivation of the sun's light, in order to determine the places at which a given lunar eclipse will be visible, it is only necessary to determine the places which will have the moon above their horizon at the time. The calculation is easily made, but for practical purposes it is sufficient to proceed as follows: Take a common terrestrial globe, determine upon it the moon's place at the commencement of the eclipse, then all places on the hemisphere lying round this point will see the commencement of the eclipse; in like manner determine the hemisphere from all places of which the termination is visible; then the whole of the eclipse will be visible from all places which are common to these two hemispheres.

*Solar Eclipse.*

75. The calculation of a solar eclipse is more difficult than that of a lunar, because the moon by its interposition between the earth and sun will intercept the sun's light from some portions of the earth's surface, but not from others, and the sun may be visible at a given place during an eclipse although the eclipse may not be visible.

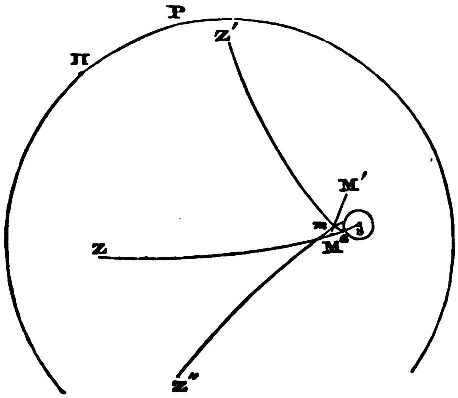
We shall attempt to give some account of the mode of calculating the circumstances of a solar eclipse, premising that as we have already supposed the earth to remain fixed and the moon to move in a relative orbit, so here we shall suppose the sun to be fixed and the moon to be apparently depressed by the difference of the solar and lunar parallax, or (as we may call it) the *relative* parallax. This relative parallax, as well as the apparent diameters of the sun and moon, are known from the Nautical Almanack, or some equivalent work.

76. Let $P$ be the pole of the equator, and for distinctness' sake, let the plane of the paper be the plane of the solstitial colure: $S$ the sun's centre, supposed fixed, $MM'$ the moon's relative orbit; round $S$ draw a small circle, having for its radius the sum of the apparent semidiameters of the sun and moon; then to any place, which is so situated that the moon's centre can be sufficiently depressed by parallax to make it fall within this small circle, the eclipse will be visible.

Let us find the place at which the eclipse will be first seen. Draw $SaM$ an arc of a great circle, and make $Ma$ equal to the relative parallax, and produce $SM$ to $Z$, so that $Za = 90^0$, then $Z$ will be the zenith of the place required, for to such a place $M$ will be depressed towards $S$ by the whole relative horizontal parallax.

Again, suppose we wish to determine those places on the earth's surface to which the first contact for different positions of the moon first becomes visible. Let $m$ be any position of the moon in the relative orbit, then from $m$ we can draw, in general, two arcs of great circles each equal to the relative horizontal parallax to meet the small circle round $S$, and if

we produce these to $Z'$ and $Z''$, making $mZ' = mZ'' = 90^0 -$ the relative parallax, the moon will appear depressed as much




as possible to these two places, and therefore for the position $m$ of the moon $Z'Z''$ are the zeniths of the two places to which an apparent contact will be first visible. And so we may trace a curve on the earth's surface including all places determined by this construction.

In like manner any other problem may be solved; the most general perhaps is this, To find all places on the earth's surface at which a given portion of the sun's surface will appear obscured at a given time.

By methods founded upon the principles which we have endeavoured briefly to describe, maps are constructed, such as those in the Nautical Almanack, exhibiting curves on the earth's surface comprehending all places for which an eclipse will be visible in a given degree.

## ON THE SATELLITES OF THE PLANETS.

77. The satellites have been already described as secondary bodies, which attend the planets in their orbits, and revolve about them in the same manner as the planets themselves revolve about the sun. The earth is, as we well know, attended by one such satellite, viz. the moon, a body which

on account of its peculiar interest to ourselves has already received a separate notice. In these systems Kepler's laws are obeyed, except so far as the motion is disturbed by extraneous causes.

78. The satellites of Jupiter are those which have been most observed; they revolve from west to east, that is, in the same direction as the moon and planets, and in planes nearly coinciding with the equator of the planet. Their eclipses are frequent; in fact, on account of the smallness of the inclinations of their orbits and the great length of Jupiter's shadow, three out of the four are eclipsed at every revolution, usually the fourth also, though on account of the greater inclination of its orbit not invariably.

79. The most remarkable result of observations of Jupiter's satellites is the discovery of the fact of the propagation of light with a finite velocity, and the actual calculation of that velocity. This we proceed to explain.

It was observed by Roemer, that the eclipses of Jupiter's satellites always happened too *soon*, that is, sooner than he expected from an average of observations, when Jupiter was in geocentric *opposition*, and therefore the earth as near to Jupiter as possible, and too *late* when in geocentric *conjunction*, and therefore the earth as far from Jupiter as possible. This was accounted for by the hypothesis that light required a finite time for its propagation, and that therefore the same phænomenon would not appear to happen at the same moment to two observers at stations 190,000,000 miles apart, as is in fact the case with two observers at opposite sides of the earth's orbit. It is easy to see that, supposing the error in the time of the eclipses to be due to this cause, we have sufficient data for calculating the actual velocity of light, and the velocity thus found coincides with that which results from the phænomenon of aberration, which has been already explained (Art. 49): the coincidence of the two independent determinations leaves no doubt respecting the accuracy of each.

## ON TIME.

80. There are three kinds of days recognized by astronomers, viz. Sidereal, Solar, and Mean Solar. Each of them is divided into 24 hours, each hour into 60 minutes, and each minute into 60 seconds.

The *sidereal* day is the interval between two successive transits of the true first point of Aries; or it is the time of the earth's revolution on its axis. Sidereal hours are those marked by the sidereal clock already described. (Art. 28.)

The *solar* day is the interval between two successive transits of the sun, and is therefore not of constant length; hence, although the beginning of this day is marked by a very obvious phænomenon, viz., the transit of the sun, yet it would be extremely inconvenient as the ordinary standard of time.

The *mean solar* day is equal in length to the average of true solar days; its commencement is marked by no actual phænomenon, but by the transit of an imaginary sun which is supposed to move uniformly in the equator with the sun's *mean* or *average* velocity.

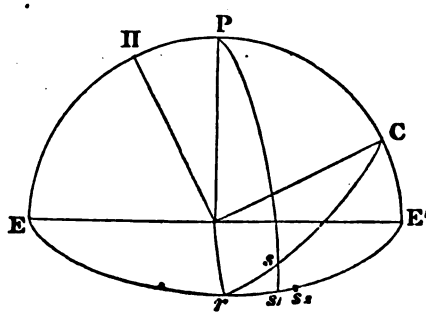
Ordinary clocks indicate mean solar time, and are therefore sometimes in advance of the sun, sometimes behind it. The quantity which must be added to, or subtracted from, true solar, to give mean solar time, is called the equation of time, and requires a particular explanation.

81. *The Equation of Time.*

The equation of time may be conceived of as arising from two distinct causes, viz., first, the motion of the sun in the ecliptic, and not in the equator, in consequence of which the time of noon would vary even if the motion of the sun in its orbit were uniform; and, secondly, the variable motion of the sun in its orbit, in consequence of which there would be an equation of time even though the ecliptic had no obliquity. It will give distinctness to our explanation to consider these two parts of the equation separately.

I. Let $E \Upsilon E'$ be the equator, $\Upsilon C$ the ecliptic, $P$, $\Pi$ their respective poles: and let $S$ be the sun in the ecliptic between $\Upsilon$ and the summer solstice. Draw the declination

circle $PSS_1$, then $S_1$ is the place of the sun in the equator. Now the *mean* sun is to be supposed to start with the true sun from ♈, and to move in the equator with the sun's velo-




city; hence it is manifest that the two suns will be on the solstitial colure at the same time, and will meet at the autumnal equinox. But between ♈ and the solstice, the R.A. of the mean will be greater than that of the true sun, for the R.A. of the mean sun is to be equal to $S$♈, which it is easy to see is greater than $S_1$♈; because, if we take the case of the sun being very near ♈, we may treat $S$♈$S_1$ as a plane right-angled triangle, in which the hypothenuse $S$♈ is the greater side, and the same will be true until the sun reaches the solstice, when the R.A. of the two is the same. Take then $S_2$ as the place of the mean sun, $S_2$ being further from ♈ than $S_1$. Now the earth revolves about its axis in the same direction as that in which the sun moves, consequently the meridian of any place is brought to $S_1$ before it reaches $S_2$, that is, *apparent* noon precedes *mean* noon when the sun is moving from an equinox to a solstice. In like manner it will appear, that the reverse is the case from a solstice to an equinox. Hence the equation of time, so far as it depends upon the obliquity of the ecliptic, is *subtractive* from an equinox to a solstice, *additive* from a solstice to an equinox, and vanishes at both solstice and equinox.

II. In consequence of the excentricity of the sun's orbit, and the law of Kepler, according to which the areas swept out in equal times are equal, the sun moves with more

than its mean velocity when near perigee, and less when near apogee. Suppose a fictitious sun to move in the ecliptic with the sun's mean motion, and to coincide with the real sun in perigee; then from perigee to apogee the real sun will be before the mean sun, and therefore *mean* noon will precede *apparent*; the opposite will be the case when the sun is moving from apogee to perigee: hence from perigee to apogee the equation of time is *additive*, from apogee to perigee *subtractive*, and zero at both apogee and perigee.

To conceive of these two effects as combined, it is only necessary to consider the sun which moves in the ecliptic, and which we spoke of as the real sun when discussing the equation of time arising from the obliquity, to be not the real sun, but a second fictitious sun moving in the ecliptic with the sun's mean motion, and coinciding with the real in perigee and apogee.

82. *To find the Mean Time by observation.*

If we have a fixed Observatory, nothing is more simple, because we have only to observe the time of the sun's transit, which will give the time of apparent noon, from which we deduce the time of mean noon by adding or subtracting the equation of time: or we may observe the transit of a known star, which will give us the sidereal time, from which the solar may be deduced; for the sidereal time determines the position of ♈, and since we have tables giving the mean R.A. of the sun, we may hence deduce the position of the mean sun with respect to our meridian at the time of observation.

But when we have no fixed Observatory, we must resort to other methods: we shall describe only one. Let an observation of the altitude of the sun be made before noon, and another made when the altitude is the same after noon; then the sun was on the meridian at a period equidistant from the two observations; or, if a clock indicates $h$ and $h'$ hours at the two observations respectively, the time of apparent noon by the clock is $\frac{h+h'}{2}$: apparent noon being known, the mean time may be deduced.

The preceding result is only approximately true, on account of the sun's motion between the two observations; but, if necessary, a correction may be introduced.

83. *On the different kinds of Years.*

Three kinds of years may be reckoned, which we may call astronomical, the *tropical*, the *sidereal*, and the *anomalistic*; besides which there is the year in the vulgar acceptation of the word, which we may call the *civil* year.

The *tropical* year is the interval between the two successive arrivals of the sun at the first point of Aries, and its length is about $365^d\ 5^h\ 48^m$.

The *sidereal* year is the interval between the sun's leaving and returning to the same point of the heavens. Its length differs from that of the tropical year on account of the motion of ♈, which has been before explained, and is found to be $365^d\ 6^h\ 9^m$.

The *anomalistic* year is the interval between two successive passages of the sun through perigee, and differs from the preceding on account of a slow motion of the perigee, produced by the disturbance of the planets; its length is $365^d\ 6^h\ 13^m$.

No one of these years would be convenient for ordinary purposes, because a year, to be convenient, should consist of an integral number of days. The *civil* year is therefore made to consist of 365 days, but to prevent the seasons from falling at different periods of the year, every fourth year, or leap year, is made to consist of 366 days. By this means the average length of a civil year is made to be $365^d\ 6^h$, which is rather more than a tropical year; to correct this error, which in the course of centuries would become considerable, the intercalation is ordered to be omitted in the years completing centuries, when the number of centuries is not divisible by four; thus only 97 days are intercalated in 400 years instead of 100: the error which remains will not amount to a whole day in 4500 years*.

* On the subject of the Calendar, and the different modes of fixing dates adopted in various ages and various countries, the student is referred to Sir Harris Nicolas' *Chronology of History*, being a volume of the *Cabinet Cyclopædia*.

## ON THE MODES OF DETERMINING TERRESTRIAL LONGITUDE.

84. The problem of finding the longitude is one of the greatest practical importance, and one which for a long time presented great difficulties. It is evident that if by any means we can ascertain at a given moment what is our own mean time of day, and also what is the mean time at some given place, as Greenwich, we shall know how many degrees we are situated to the east or west of Greenwich; now the time of day at any place may be ascertained by observation, as already explained (Art. 82), hence the problem of the longitude reduces itself to that of ascertaining at any given time and place mean Greenwich time.

It is obvious therefore that the problem is solved, if we can obtain watches set to Greenwich time and of sufficient accuracy to be depended upon; and the perfection to which chronometers have been brought, has in fact made the discovery of the longitude at sea a matter of perfect simplicity. Nevertheless there are certain astronomical methods which deserve notice; the following are some of them.

85. *To find the longitude by observation of an eclipse of one of Jupiter's satellites.*

The time of an eclipse of one of Jupiter's satellites is evidently quite independent of the place from which it is observed; and the motions of the satellites being known, it is possible to predict every eclipse and to register the time of its happening according to Greenwich mean time; if then an observer at another place observes an eclipse and notes the time of its happening, he will be able to compare, by means of the Nautical Almanack, his own mean time with that of Greenwich, and so determine his longitude.

86. *To find the longitude by observing the distance of the moon's centre from certain fixed stars.*

This is a mode which can be practised at sea.

Let the observer with a Hadley's sextant observe the distance of a star from the moon's centre, by noting its distance

from the nearest and furthest point of the moon's disk and taking half the sum of these distances. The distance thus found must be corrected for parallax and refraction.

Also, let the observer, immediately after making the preceding observation, (or another observer simultaneously,) take the altitude of the star. From this observation, coupled with the N.P.D. of the star and the latitude, which are supposed known, the apparent time at the place of observation may be found.

Now the Nautical Almanack gives the distance of the moon from certain stars, of which that observed must be one, for every three hours, and from this we can easily determine the Greenwich time for which the distance of the moon and star is exactly that which we have determined, that is, we can determine the Greenwich time of taking the observation; and the comparison of this with the apparent time at the place of observation determines the longitude.

87. The longitude of a fixed Observatory may be determined thus. Suppose the moon is observed at the Observatory in question to culminate with a certain fixed star, then the same star and the moon will not culminate together to a place on a different meridian, because the moon will have moved in R.A.: suppose that the difference of time ($a$) between the transit of the moon and the star is observed at the second place, which will be in fact the moon's motion in R.A. Also let $A$ be the motion of the moon in R.A. in the time of a complete revolution of the earth, and $x$ the difference of longitude of the two places, then we shall have

$$x^0 : 360^0 :: a : A;$$

or $x^0 = \frac{a}{A} 360^0$, the difference of longitude required.

ON COMETS.

88. We shall devote only a few words to comets. The[illegible] are in fact planets revolving in conic sections about the s[illegible] in their fo[illegible] but in orbits of very great excentricity; t[illegible] orbits of [illegible] number are ellipses so much elongat[illegible] as to [illegible] parabolas, and some but not many move [illegible]

hyperbolas. The number of these bodies belonging to, or rather infesting, our system, appears to be very great indeed; but few have been sufficiently observed to identify them, when after travelling far into space they again approach the sun, and become visible to us. Of the few so observed we may notice that which bears the name of Halley's Comet, the period of which is about 75 years; also Encke's Comet, the period of which is $3\frac{1}{2}$ years; and Biela's, $6\frac{3}{4}$ years.

The last two possess remarkable interest; the former from the fact of its period being found to diminish, or its distance from the sun to diminish, a fact which would seem to indicate the existence of some resisting medium in which the planets move, but which has at present not sensibly affected the motions of the larger planets. The latter is remarkable as a double comet, or combination of two comets, probably revolving round the centre of gravity of the two.

## GENERAL VIEW OF THE SOLAR SYSTEM.

89. We shall conclude this treatise with a general view of the Solar System, compiled from Humboldt's *Cosmos*, (Sabine's translation), a work of most extraordinary learning and scientific value, to which we would gladly direct the student's attention.

"The Solar System consists, according to our present knowledge*, of eleven principal planets, eighteen moons or satellites, and myriads of comets, three of which (called planetary comets) do not pass beyond the orbits of the principal planets. We may, with considerable probability, include within the dominion of our sun, a revolving ring of finely divided or nebulous matter, situated perhaps between the orbits of Venus and Mars, but certainly extending beyond that of the earth, which we call the Zodiacal Light; and a host of extremely small asteroids, the paths of which intersect, or very nearly approach, that of the earth, and which present to us the phænomena of aerolites or shooting stars.

"It has been proposed to consider the telescopic planets,

* Neptune and forty asteroids have been discovered since this was written.

Vesta, Juno, Ceres, and Pallas, with their more excentric intersecting and greatly inclined orbits, as forming a middle zone, or group, in our planetary system; and if we follow out this view, we shall find that the comparison of the inner group of planets, comprising Mercury, Venus, the Earth, and Mars, with the outer group consisting of Jupiter, Saturn, and Uranus, presents several striking contrasts. The planets of the inner group, which are nearer the sun, are of more moderate size, are denser, revolve about their respective axes more slowly in nearly equal periods, which differ little from twenty-four hours, are less compressed at the poles, and with one exception are without satellites. The external planets, more distant from the sun, are of much greater magnitude, five times less dense, more than twice as rapid in their rotation round their axes, more compressed at their poles, and richer in moons in the proportion of 17 to 1; if Uranus really has the six satellites ascribed to it.

"In viewing these general characteristics of the two groups, we must admit however that they cannot be strictly applied to each of the planets in particular; nor are there any constant relations between the distances of the planets from the sun, their absolute magnitudes, densities, times of rotation, excentricities, and inclinations of orbits and of axis. We find Mars, though more distant from the sun than either the Earth or Venus, inferior to them in magnitude. Saturn is less than Jupiter, and yet much larger than Uranus. The zone of the telescopic planets comes next before Jupiter, the greatest of all the planetary bodies; and yet the disks of these small planets are less than twice the size of France. Remarkable as is the small density of the planets which are furthest from the sun, yet neither in this respect can we recognise any regular succession. Uranus appears to be denser than Saturn; and we find both Venus and Mars less dense than the earth, which is situated between them. The time of rotation decreases on the whole with increasing solar distance, but yet it is greater in Mars than in the earth, and in Saturn than in Jupiter. Among all the planets, the orbits of Juno, Pallas, and Mercury, have the greatest excentricity; and Venus and the Earth which

immediately follow each other have the least; while Mercury and Venus (which are likewise neighbours) present, in this respect, the same contrast as do the four smaller planets, whose paths are so closely interwoven. The excentricities of Juno and Pallas are nearly equal, but are each three times as great as those of Ceres and Vesta. Nor is there more regularity in the inclination of the planes of the orbits of the planets to that of the ecliptic, or in the position of their axes of rotation relatively to their orbits; on which latter position the relations of climate, seasons, and length of days depend, more than on the excentricity. It is in the planets which have the most elongated ellipses, Juno, Pallas, and Mercury, that we find, though not in equal proportion, the greatest inclination of the orbits to the ecliptic. Neither do we find a regular order of succession in the position of the axes of the few planets, (four or five,) of the planes of rotation of which we have at present any certain knowledge. Judging by the position of the satellites of Uranus, the axis is inclined barely 11° to the plane of its orbit; and Saturn is placed intermediately between this planet, in which the axis of rotation almost coincides with the plane of its orbit, and Jupiter, whose axis is almost perpendicular to it.

"Among the fourteen satellites, concerning which investigation has arrived at some degree of certainty, the system of the seven satellites of Saturn offers the greatest contrasts, both of absolute magnitude, and of distance from the planet. The sixth satellite is probably but little smaller than Mars (whose diameter is twice that of our moon), while, on the other hand, the two innermost satellites belong to the smallest cosmical bodies of our system. After the sixth and seventh of the satellites of Saturn comes, in order of volume, the third and brightest of Jupiter's.

"The twelve moons attendant on Saturn, Jupiter, and the Earth, all move, as do their primary planets, from West to East, and in elliptical orbits differing little from circles. The satellites of Uranus exhibit some remarkable differences from the movements of other satellites and planets. In all other cases, the orbits are but little inclined to the ecliptic, and the

movements are from West to East, including Saturn's rings, which may be regarded as belts formed of an aggregation of satellites; but the satellites of Uranus move in planes almost perpendicular to the ecliptic, and the direction of their motion is from East to West.

"It appears highly probable, that the times of rotation of *all* secondary planets, or satellites, are the same as their times of revolution round their primaries; so that they always present to the latter the same face. In the case of the moon, owing to certain disturbing causes, rather more than half the surface is visible to us at different times; but as much as three-sevenths of the whole surface is always invisible.

"Of all planetary bodies, comets,—though their mean mass is probably much less than the five-thousandth part of the earth,—are those which occupy the greatest space, their wide-spreading tails often extending over many millions of miles. The cone of light-reflecting vapour, which radiates from them has been found in some instances to equal in length the distance of the earth from the sun. It is even probable that the vapour of the tails of the comets of 1819 and 1823 mixed with our atmosphere.

"Comets shew such diversities of form, that the description given of one of them cannot be applied without much caution to another. The fainter telescopic comets are for the most part without tails. We can distinguish in the larger comets the *head* or *nucleus*, and the *tail*. The intensity of light in the nucleus of a comet does not increase in a uniform manner towards the centre, but bright zones alternate with concentric nebulous envelopes. The tail appears sometimes single, more rarely double; in the comet of 1744, the tail had six branches. The tails are sometimes straight, sometimes curved, and sometimes inflected like a flame in motion; they are always turned from the sun, and so directed that the prolongation of the axis would pass through the centre of that body.

"Amongst the countless host of uncalculated or still undiscovered comets, it is highly probable that there are many, the major axes of whose orbits may far exceed even that of the comet of 1680, to which Encke assigns a period of

upwards of 8800 years. In order to afford some notion of the distance in space of a fixed star or other sun from the aphelion of this comet, (the one of the bodies of our solar system, which, according to our present knowledge, attains the greatest degree of remoteness,) it may be mentioned, that, according to the most recent determinations of parallax, even the nearest fixed star is at least 250 times more distant from our sun than this comet at its aphelion.

"Another class of bodies remains to be considered; namely, those minute asteroids, which, when they arrive in a fragmentary state within our atmosphere, we designate by the name of *aerolites*, or meteoric stones.

"Shooting stars, fireballs, and meteoric stones, are with great probability regarded as small masses moving with planetary velocities in space, and revolving in conic sections round the sun, in accordance with the laws of universal gravitation. These masses approach the Earth in their path, are attracted by it, and enter our atmosphere, becoming luminous at its limits, when they frequently let fall stony fragments, heated in a greater or less degree, and covered with a shining black crust. While there are exploding and smoke-emitting balls of fire, which are luminous even in the bright sunshine of a tropical day, and sometimes exceed in size the apparent diameter of the moon, there are on the other hand shooting stars which fall in immense numbers, and are of such small dimensions, that they exhibit themselves only as moving points, or as phosphorescent lines. Whether among the many luminous bodies which shoot across the sky, there may not be some of a different nature from others, still remains uncertain.

"The connection of meteoric stones with the more splendid phænomenon of fireballs, and the fact that meteoric stones sometimes fall from fireballs with a force which causes them to sink to a depth of from ten to fifteen feet into the earth, have been proved by a variety of observations. In some instances a small and very dark cloud forms suddenly in a perfectly clear sky, and the stones are hurled from it with a noise resembling repeated discharges of cannon. Such a cloud, moving over a whole district of country, has sometimes

covered it with thousands of fragments, very various in size, but similar in quality.

"We have as yet scarcely any knowledge in regard to the physical and chemical processes which contribute to the formation of these phænomena. We know by measurement the astonishing and quite planetary velocity of shooting stars, fireballs, and meteoric stones; in this respect therefore we are able to recognise what is general and uniform in the phænomena; but the successive transformations undergone are not known to us. The largest meteoric masses yet known to us are seven and seven and a half feet in length; the meteoric stone of Ægos Potamos, celebrated in antiquity, and mentioned in the chronicle of the Parian Marbles, and which fell about the year of the birth of Socrates, has been described as being of the size of two millstones and equal in weight to a waggon load.

"Shooting stars fall either singly or sporadically, or in groups of many thousands which are compared by Arabian writers to flights of locusts. The latter cases are periodical, and the meteors are then seen in streams, moving for the most part in parallel directions. Of the periodic groups those hitherto best known are the phænomena of the 12th to the 14th of November, and on the 10th of August or day of St Lawrence, whose "fiery tears" were long since recognised in England as a recurring meteorological phænomenon.

"It is probable that the different streams of meteors, each consisting of myriads of small bodies, intersect the orbit of the earth in the same way that Biela's Comet does; according to this view, we may imagine that they form a continuous ring, each pursuing its course in a common direction. We cannot yet determine whether the variations in the epochs at which the stream becomes visible to us, and the observed retardations of the phænomena, indicate a regular progression of the points of intersection of the ring with the earth's orbit, or whether they are to be explained by the irregular grouping of these very small bodies, and by the supposition that the zone formed by them has a width which the earth requires several days to traverse. Should increased probability be given to the former of these hypotheses, the discovery of

older observations of these phænomena will acquire a special interest.

"To complete our view of all that belongs to the solar system, which now, since the discovery of the small planets, of the comets of short period, and of the meteoric asteroids, appears so complex and so rich in forms, we have yet to consider the *Zodiacal Light*. Those who have dwelt long in the zone of Palms, must retain a pleasing remembrance of the mild radiance of the phænomenon, which rising pyramidally illumines a portion of the unvarying length of the tropical nights. In the obscurer sky and thicker atmosphere of the temperate zone, the Zodiacal Light is only distinctly visible in the beginning of Spring, when it may be seen after evening twilight above the Western horizon, and at the end of Autumn, before the commencement of morning twilight above the Eastern horizon.

"We may with great probability attribute the Zodiacal Light to the existence of an extremely oblate ring of nebulous matter, revolving freely in space between the orbits of Venus and Mars. We can indeed at present form no certain judgment concerning the true dimensions of the supposed ring; its possible augmentation by emanations from the tails of many millions of comets when at their perihelia; the singular variability of its extent, which seems sometimes not to exceed that of our own orbit; or concerning its not improbable intimate connection with the more condensed cosmical vapour in the vicinity of the sun. The nebulous particles of which the ring consists, and which revolve round the sun according to the same laws as the planets, may be either self-luminous, or may reflect the light of the sun."

THE END.

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